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ABSTRACT

This second volume in the Topics for Teachers series provides a wide range of activities designed to be both useful for teachers and informative for students. The investigations contained in this document are representative of some of the embodiments of important mathematical principles in the physical world. The activities included were chosen according to the following considerations: (1) the activity involves one or more important concepts of secondary school mathematics; (2) the investigation is typical of experiments commonly done in junior or senior high science laboratories; (3) the activity involves simple, easily obtainable equipment; and (4) the investigation is manageable within ordinary school time limits. The experiments are grouped into the following categories: (1) functions; (2) measurement; (3) ratio and proportion; (4) spatial relationships; and (5) modeling, predicting, and decision making. Each section concludes with an extensive set of teaching notes and suggestions for other activities and applications. (MP)

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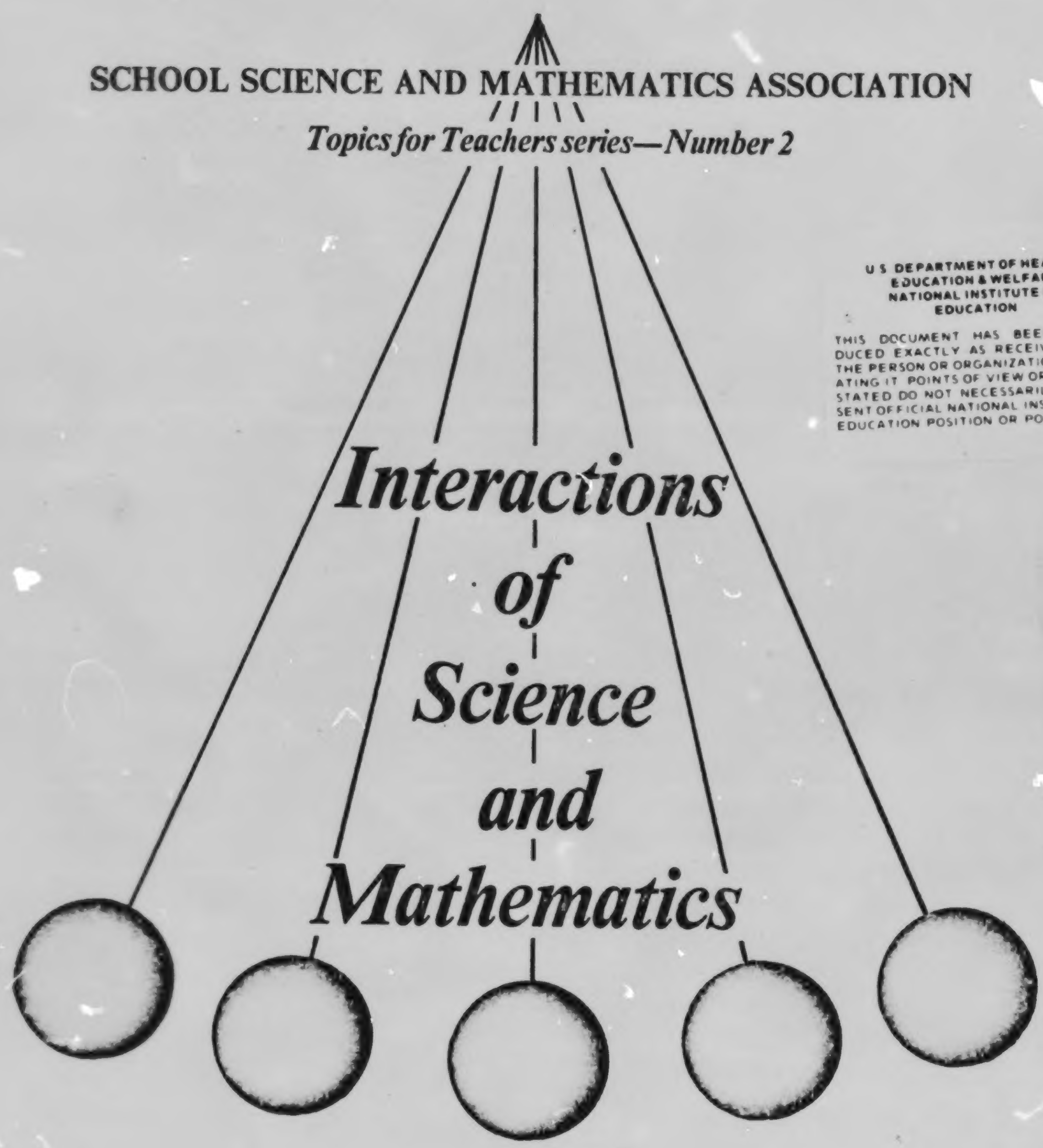
Topics for Teachers series—Number 2



Interactions of Science and Mathematics

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By
Peggy A. House

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SE 033 191

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION

Topics for Teachers Series

Number 2

INTERACTIONS OF SCIENCE AND MATHEMATICS

A Set of Activities

Peggy A. House
University of Minnesota

April 1980

ERIC® Clearinghouse for Science, Mathematics
and Environmental Education
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FOREWORD

The ERIC Clearinghouse for Science, Mathematics and Environmental Education is pleased to publish this second volume in the Topics for Teachers series of the School Science and Mathematics Association. Quite obviously, it focuses on two of the subject areas of concern to this Clearinghouse, providing a range of activities that should prove useful for teachers and informative for students. We appreciate the work of Peggy A. House in preparing this publication, and thank Sujata Seam for providing the drawings. The manuscript was reviewed by members of the Publications Committee of the School Science and Mathematics Association, and we also thank them for their help: Michael Agin, Michigan Technological University; James P. Barufaldi, University of Texas at Austin; and Gerald Krockover, Purdue University.

We are pleased to make this publication available to teachers and students: may it provide many hours of stimulating enjoyment!

Marilyn N. Suydam

Associate Director for
Mathematics Education,
ERIC/SMEAC

Chairperson, Publications
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Chapter 1

Introduction

Mathematics is a study of patterns. The patterns of mathematics can arise from a wide range of real phenomena, or they can be the products of human creativity. Unfortunately, this broader view of mathematics as a study of patterns often eludes pupils, because they have developed a much more atomistic view of mathematics as a collection of very specific rules, formulas, and theorems.

The physical world is rich in patterns and relationships which embody important mathematical principles. This book suggests some of the many relationships or applications which occur in the physical world. In particular, it will focus on:

patterns which we observe in selected physical situations;

variables and other mathematical symbols which we use to represent patterns;

functions which describe the relationships among quantities or objects -- these include both *numerical* and *spatial* relationships;

measurement and the gathering, organizing, communicating, and using of data;

models or abstractions which enable us to explain, to predict, and to make decisions.

The author hopes to stress that "applications" are more than just a "laying on" (i.e., *apply*-ing) of already learned concepts and principles. Rather, *applications* will be thought of as *embodiments* -- realities in the physical world which involve certain relationships or patterns in which we may be interested. Readers are asked to examine these embodiments as representative of phenomena we could study as a *source* for deriving mathematical ideas, to look for patterns among the patterns themselves, and to classify relationships according to certain common characteristics. However, before discussing further the contents and objectives of the chapters which follow, let us consider briefly the rationale for this approach.

Rationale

A perusal of mathematics curriculum materials at all levels soon reveals that the concept of *function* permeates virtually all of school mathematics. Closely related to an understanding of functions are the concepts of variability and proportionality. These ideas are essential to the recognition and study of patterns and to the prediction of changes and expected outcomes. Not only in mathematics, but in science too, patterns, relationships, variability, and functions are key concepts in understanding, interpreting, and predicting physical phenomena. The relationships we encounter in the physical world are rendered more understandable when represented in the language of mathematics, while, on the other hand, the mathematical relationships in which we frequently are most interested are those which arise from real applications.

We must keep in mind, however, that functions, proportional reasoning, and abstract symbolic representations require the Piagetian stage of formal operational thought, and this has serious implications for the teacher of junior and senior high school mathematics and science. While Piagetian research has contributed significantly to the understanding of how children learn, it has also led to a common misinterpretation. It is unfortunate that the age of twelve years is usually written in parentheses after the heading "Formal Operations," for it has led many teachers to the mistaken conclusion that the ability to reason abstractly is the norm for junior high school pupils and older students. On the contrary, while formal operations may begin to emerge at approximately the age of junior high school pupils, the development of logical thought is gradual and evolutionary, and many, perhaps most, pupils remain primarily concrete operational during much of their secondary school experience. It is essential, then, that secondary school pupils as well as younger children experience adequate interaction with concrete embodiments of mathematical concepts through direct action on physical objects.

Most educators accept as axiomatic that understanding and meaningfulness are rarely, if ever, all-or-nothing insights either in the sense of being achieved instantaneously or in the sense of embracing the whole of a concept and its implications at any one time. Rather, any sudden perception or flash of insight comes only to those who have struggled to extend or apply concepts which they have gradually and partially understood earlier. Therefore, teachers must provide pupils continually with recurring but varied contacts with fundamental ideas and processes which they hope pupils will develop.

Polya[1] stressed the importance of three phases of learning: First, an exploratory phase of action and perception which moves on an intuitive, heuristic level; second, a formalizing phase which ascends to a conceptual level and which introduces terminology, definitions, and proof; and, finally, an assimilation phase in which the learning is absorbed into the integral mental structure of the learner.

Polya's description of successive stages, together with the recognition of the cognitive developmental level of the adolescent, have direct implications for the secondary mathematics classroom, especially when one is concerned with an abstract concept like function. In particular, they suggest the importance of physical embodiments as one source for the development as well as for the application of such concepts.

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[1] Polya, George. "On Learning, Teaching, and Learning Teaching." American Mathematical Monthly, 70 (June-July 1963), 605-619.

Much of the time, however, the term "applications" connotes a kind of "laying on" which, in the case of ideas, suggests a putting to use of the ideas after they are developed. The "applications" which we speak of here are, rather, "embodiments," since this term suggests a broader understanding which encompasses the specifics from which ideas emerge as well as the uses to which these ideas are put. It is the fundamental premise of this book that the learning of concepts such as function, proportionality, and modeling is an interactive process which begins with the learner's action upon concrete embodiments, proceeds to abstraction, and returns again to new embodiments as the source of further refinement and generalization. For this process to take place, the learner must encounter these concepts in a wide variety of embodiments from which eventually one abstracts the essential ideas.

Goals and Objectives

The investigations in the following chapters are intended for secondary school teachers of both mathematics and science. They are assembled from among many appropriate activities as representative of some of the embodiments which can be used as a basis for developing mathematical ideas. The activities included here were chosen according to the following considerations:

- The activity involves one or more important concepts of junior or senior high school mathematics.
- The activity is typical of experiments commonly done in junior or senior high school science labs.
- The activity involves simple equipment which you should find easy to assemble yourself or to obtain from your school's science department.
- The activity is manageable within ordinary school time limits.

The investigations are written for the teacher, but it is hoped that they are presented in a form which can easily be adapted to the needs, maturity, and background of the teacher's own pupils. Teachers should carry out the investigations themselves before adapting them for class use. The teaching notes in each chapter are intended to help teachers implement the investigations. Each chapter also contains suggestions of other embodiments which might be developed into similar investigations.

Throughout the book readers will encounter more questions than answers. By seeking the answers and by generating still more questions, teachers themselves should come to a deeper insight into the interactions of science and mathematics. These insights, transmitted to students, can go far to develop that broader vision of the patterns and relationships which are science and mathematics. Thus, the investigations suggested here are for every science teacher whose students have moaned about the mathematics involved, and for every mathematics teacher whose students have asked, "Why do we need to know this?"

Chapter 2

Functions

Few concepts are as fundamental in school mathematics as the concept of a function, and few, perhaps, are as misunderstood or, more frequently, as incompletely understood. For this reason, it is useful to consider the role of applications in the development and generalization of the concept of function.

A number of tasks are associated with the development of the function concept. These include recognizing a function, predicting the range given the domain and a rule, predicting or deducing the rule given a set of ordered pairs, recognizing invariants (identities), and recognizing inverses. One approach to developing pupils' abilities to perform these and related tasks is through laboratory activities in which the pupil investigates concrete problems where a relationship exists between two quantities and where variation in one of these brings about a predictable variation in the other.

In these activities, the pupils are expected to make determinations on which quantities to vary and which to control; make observations; take measurements; graph, tabulate, or otherwise record data; propose and test hypotheses; deduce the specific functional relationship of each embodiment; make predictions of future results; verify predictions; and relate outcomes to other experiments.

The chapter first describes a series of investigations for students. The teaching notes which follow point out important concepts for consideration as well as relationships among the investigations. The chapter concludes with suggestions of other applications which could be developed into similar investigations for classroom use.

Investigations

Investigation 2.1

Use tinker toys to make a "cart" such as the one shown in Figure 2.1. Cut a strip of cardboard and lay it on top of the wheels of the cart. Now roll the cart through a distance d and measure both the distance traveled by the cart and the distance traveled by the cardboard. Repeat the experiment for ten different values of d. Record your results in a table and on a graph. Try to explain what is happening and why. Also, express the relationship in a mathematical statement. Would the same results hold if the cart had larger wheels? If the cart had more than four wheels? If the cardboard was placed on the axle instead of on the wheels? If the axle was longer?

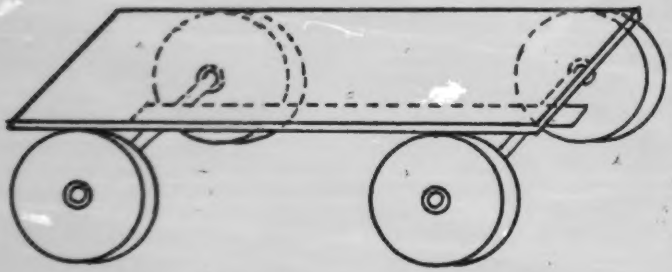


Figure 2.1

Investigation 2.2

Take 20 different round objects (jar lids, buttons, tin cans, balls, etc.) and measure the diameter and circumference of each. Organize your data in a table, and represent it on a graph. Describe the pattern you see. Express the pattern in a mathematical symbolism. Do you get the same result from circles (2-dimensional) as from spheres (3-dimensional)? Do all the points "fit" exactly on the graph? If not, where is the deviation the greatest? Are there some reasons that might account for this?

Investigation 2.3

Hang a spring from a hook. Measure the length of the spring. Now add one weight (one washer) and again measure the length of the spring. Continue to add one washer at a time until you have 10 or 12 sets of data. (Stop, however, if the spring nears its elastic limit.) Do the entire experiment two more times using different springs. Try to select springs that appear to have different stiffness. Organize all of your data, first in a table and then on a graph. What do the results tell you about the relationships between weight and stretch? What are the variables in this situation? What are the constants?

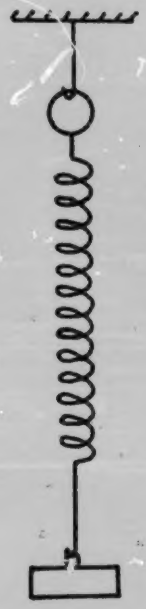


Figure 2.2

Investigation 2.4

Clamp a metal strip (a hacksaw blade, for example) to the table as shown in Figure 2.3. Place a block of wood above the blade to assure that the blade bends at the table edge. Mount a vertical scale behind the blade. Attach a wire or string to the blade so that the weights hung from the spring at A pull the blade downward. Pass the other end of the string around a pulley hung directly above the blade so that weights hung from this string at B pull the blade upward. Add weights at A and read the position of the blade. Do this for six or more different weights. Then repeat using the same set of weights hung at B. Record the data in a table and also plot the graph.

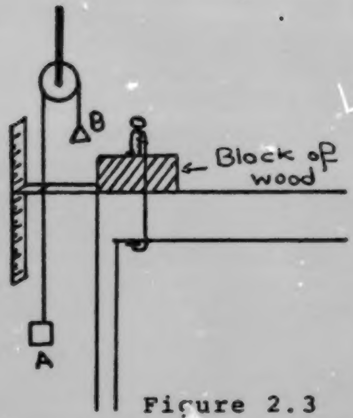


Figure 2.3

Does a given weight hung at A produce the same deflection as the same weight hung at B? How can you distinguish between deflections downward and deflections upward? How does this affect the graph? Why is it important that the pulley be directly above the point where the string is attached to the blade? Would the results change if you used a thicker blade? A longer or shorter blade? A blade of a different material?

Investigation 2.5

(The following should be done using two uncalibrated thermometers. However, since most classrooms probably do not have equipment for this, it is described here as a simulation of the actual activity.)

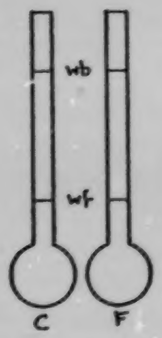


Figure 2.4

a. Make an enlargement of the two thermometers pictured in Figure 2.4. Make each drawing at least 25 cm tall.

b. Assume that you have performed the following experiment:

   (1) Two identical thermometers were placed in a pan of melting ice. (Both ice and water were present.) The mercury level in each thermometer was marked. This is the line wf, the temperature at which water freezes. On thermometer C this line was called 0 (zero); on thermometer F it was called 32.

   (2) Next both thermometers were placed in a pot of boiling water, and the mercury level again was marked. This is the line wb, the temperature at which water boils. On thermometer C it was named 100; on thermometer F it was called 212.

c. Be sure the length between wb and wf is the same for both thermometer models. Using the numbers assigned to the two points, divide the distance between them into equal segments (called "degrees"). How many degrees will be on each thermometer? Number at least 20 of the degrees on each thermometer.

d. Use the two models to determine at least 6 pairs of points (C,F) which represent equal temperatures other than (0,32) and (100,212). Graph these points; also graph the two fixed points (wf and wb).

e. Explain the relationship shown on the graphs. What temperature will C show when F is at zero? What will F read when C reads zero? Find an expression to describe the relationship between the two graphs. Show how the graphs can be used to convert from one thermometer to the other. Is there ever a temperature where the two thermometers show the same reading? If the temperature on F goes up 10 degrees, will C go up more degrees or fewer degrees or the same number of degrees? What reading on each thermometer should suggest that:

- -- you will need your snowmobile clothes
- -- you had better take a sweater
- -- you will probably be comfortable in your shirtsleeves
- -- a jump in the pool would feel good

-- you could bake a pizza

-- a cup of coffee is just right for drinking

Why is it necessary to be sure the dish of melting ice contains both ice and water? Are the same precautions necessary for the boiling point? If the ice is melting, why do we call it the freezing point?

Investigation 2.6

If a gas is held at a constant temperature while the pressure is increased, the volume of the gas decreases; if the pressure is decreased, the volume increases. Below are typical data from two experiments:

Experiment one:

Pressure (mm of Hg)	Volume (ml)
720	600
540	800
480	900
360	1200
300	1440
240	1800
180	2400

Experiment two:

Pressure (atm)	Volume (l)
0.1	10
0.2	5
0.4	2.5
0.5	2
0.8	1.25
1.0	1

Graph the data above to show volume as a function of pressure. How do your graphs compare with the graphs in the previous investigations? Can you write a mathematical expression to describe the relation of pressure and volume of a gas at constant temperature?

Investigation 2.7

Suppose you wanted to make a trip of 720 miles and you had available the following alternative modes of transportation:

vehicle	average speed
plane	360 mph
train	120 mph
car A	60 mph
car B	40 mph
motorcycle	30 mph
bicycle	6 mph

Make a table of the average speed and time needed for the trip for each alternative. Graph the data (average speed, time) for the 720 miles. Describe your graph in a mathematical expression. Is it similar to any of the previous investigations? From the graph can you determine how long the trip would take if you could average 55 mph? 100 mph? How fast would you have to travel to make the trip in 5 hours? In 15 hours?

Investigation 2.8

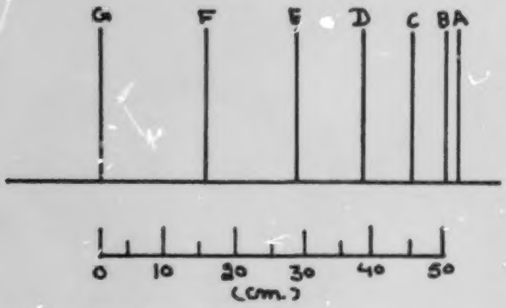


Figure 2.5

A physics student took stroboscopic pictures of a cart as it rolled along the table top. Attached to the cart was a mast whose position at each successive picture is shown in Figure 2.5. The time interval between pictures is 1/20 second.

(a) If the cart was moving from right to left, would it be gaining or losing speed? What if it moved from left to right?

(b) Graph distance vs. time for the cart's motion. Assume it moved from right to left. What does the graph indicate?

(c) Suppose we wanted to estimate the speed of the cart at a given time, say at point B. One way to approximate the speed at B is to calculate the average speed in an interval containing B, say the interval AC. Use this method to approximate the speed at positions B through F. Keep track of your data on a table like this:

Position	Interval	Distance(Δd)	Time(Δt)	Average Speed $\frac{(\Delta d)}{(\Delta t)}$
B	AC	7 cm	.10 sec	70 cm/sec
C	BD			
D				
E				
F				

(d) Plot the above data on a graph of speed vs. time. Compare the results to the graph of distance vs. time. What does each graph tell you? Is there some way you can use the graph of speed vs. time to tell you how far (distance) the cart traveled in a given time? If the cart did not accelerate at a constant rate but rather increased its acceleration uniformly, how would this show on the graph? What would happen to your measurement of speed at each point above? Can you safely assume that the speed at point C is equal to the average speed from B to D? Is this the same as the average speed from A to E? Check and see.

(e) Suppose your car's odometer breaks down while you are driving through the desert. How can you determine how far you traveled?

(f) A woman drives from home to work at the rate of 30 mph. Later she returns at the rate of 60 mph driving over the same route. What is her average speed for the round trip?

Investigation 2.9

Figure 2.6 is the representation of a strobe picture taken of a falling ball. Graph distance and speed as before, and determine whether the acceleration of the falling ball is uniform. If it is uniform, measure it.

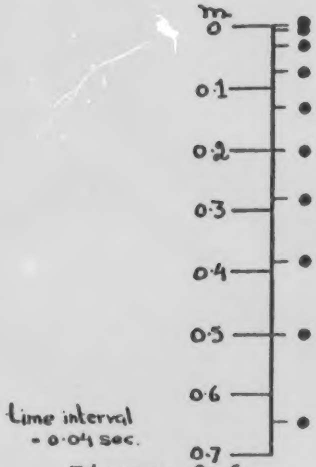


Figure 2.6

Investigation 2.10

Time (sec)	Speed (m/sec)	Time (sec)	Speed (m/sec)
0.0	0.0	6.0	27.3
1.0	6.3	7.0	29.5
2.0	11.6	8.0	31.3
3.0	16.5	9.0	33.1
4.0	20.5	10.0	34.9 (or 78 mph)
5.0	24.1		

The data above are the instantaneous speeds in a test run of a car, starting from rest, at intervals of 1.0 sec. Plot the speed vs. time graph. Use the graph in answering the following: How fast is the car going at t = 2.4 sec? How far does the car go in the time interval between t_1 = 2.4 sec and t_2 = 5.2 sec? What is the maximum acceleration of the car? When did the car reach that acceleration? What is the average acceleration during the 10 seconds? What total distance did the car travel in those 10 seconds?

Investigation 2.11

When a ball rolls across a table top and off the edge, it leaves the table with two motions: a constant velocity motion in the horizontal direction (due to the rolling) and a uniformly accelerated motion in the vertical direction (due to falling).

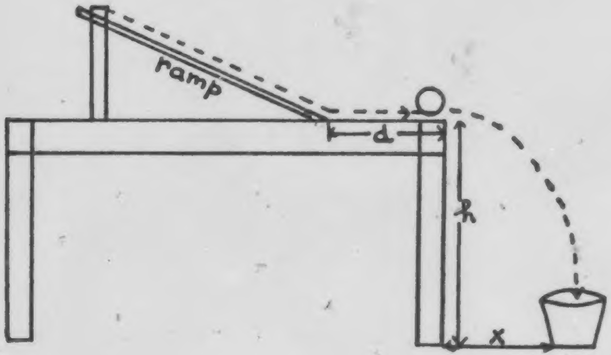


Figure 2.7

If you roll a ball down a ramp (see Figure 2.7), it will roll along the table top with a velocity which depends on the height from which it is released. You can calculate that velocity (v_o) by measuring the time (t) which it takes for the ball to roll a distance (d) along the table. This velocity, v_o, is the horizontal velocity of the ball as it leaves the table. Repeat the measurements several times, always releasing the ball from the same point, and calculate the average value of v_o.

Now, if the table has a height, h, then the instant the ball leaves the table it begins to fall the distance, h. Since the ball falls freely, you can calculate the time needed to fall to the floor from the relationship

$$h = 1/2\ a_g\ t^2$$

where a_g is the acceleration of gravity (a_g = 980 cm/sec^2 or a_g = 32 ft/sec^2).

During this time, t, the ball continues to move horizontally with the velocity, v_o. The distance it will travel is

$$x = v_o t$$

Use these relationships to predict the point where the ball will hit the floor. Test your prediction by placing a cup at that point. If your prediction is accurate, the ball should fall into the cup.

Repeat the investigation for several values of v_o. How can you change the value of v_o? If v_o increases, will the cup be closer to the table or farther from the table? Why? If you moved your experiment to a higher table, would this affect the position of the cup? How? Why? Do the outcomes change if you use a heavier ball? A bigger ball? How and why do they change?

Investigation 2.12

On an index card, make a scale drawing of an object such as a ball falling freely. Choose your scale so that about 10 or 12 successive positions of the object fit on the card. Now, using the scale, transfer the drawings to separate cards -- one position per card. Place the cards in proper sequence and staple them at one end. Holding the stapled end, flip through the book with your other hand. Does the object appear to move? Suggest some other motions you could describe in a motion book. Try a couple.

Investigation 2.13

What determines the length of time for a pendulum to make one complete swing (known as the period of the pendulum)?

Investigation 2.14

What determines the period of an oscillating spring?

Investigation 2.15

The proton and the electron in a hydrogen atom each have a charge of one unit, but the charges are opposite: The proton has positive charge, the electron negative. (It is customary to use the symbol e for the unit charge; hence, the proton and the electron have charges of ^+e and

^-e, respectively. The value of e is 1.6 x 10^{-19} coulombs.) In the hydrogen atom, the distance between the proton and the electron is about 5.3 (10^{-11}) meters, and the electrical force between the proton and the electron is found to be 8.1 (10^{-8}) newtons. As the electron and the proton are moved farther apart, the force of attraction between them decreases. The following data describe these forces:

distance between ^+e and ^-e (meters)	force (newtons)
5.3 (10^{-11})	8.1 (10^{-8})
10.6 (10^{-11})	2.0 (10^{-8})
15.9 (10^{-11})	.91 (10^{-8})
21.2 (10^{-11})	.51 (10^{-8})
26.5 (10^{-11})	.33 (10^{-8})
31.8 (10^{-11})	.23 (10^{-8})
37.1 (10^{-11})	.17 (10^{-8})
42.4 (10^{-11})	.13 (10^{-8})
47.7 (10^{-11})	.10 (10^{-8})
53.0 (10^{-11})	.08 (10^{-8})

Investigation 2.16

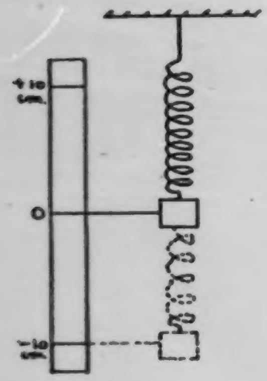


Figure 2.8

A weight hanging from a spring is at the zero mark of a vertical scale, as shown in Figure 2.8. The weight is pulled downward a distance of 10 cm and released. While the spring oscillates, a stroboscopic picture is taken with a time interval of 0.2 sec. A sample of the resulting picture is shown in Figure 2.9.

From the picture you can tell that the weight is moving faster at some places than at others. Where is its movement the fastest? Where is it slowest?

Several such pictures were analyzed and the following measurements were recorded:

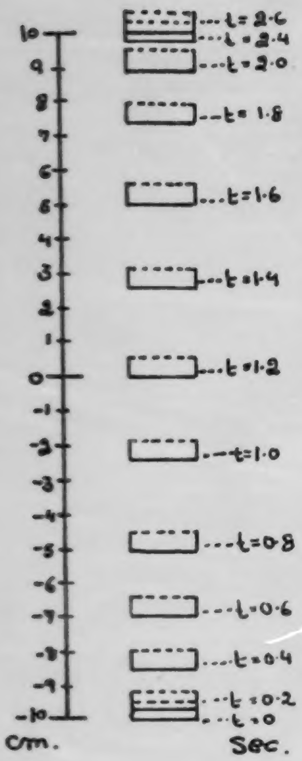

10
9
8
7
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
cm.
t=2.6
t=2.4
t=2.0
t=1.8
t=1.6
t=1.4
t=1.2
t=1.0
t=0.8
t=0.6
t=0.4
t=0.2
t=0
Sec.

Figure 2.9

time (sec)	displacement (cm)
0	-10.0
0.4	- 8.7
0.8	- 5.0
1.2	0
1.6	+ 5.0
2.0	+ 8.7
2.4	+10.0
2.8	+ 8.7
3.2	+ 5.0
3.6	0
4.0	- 5.0
4.4	- 8.7
4.8	-10.0
5.2	- 8.7
5.6	- 5.0
6.0	0
6.4	+ 5.0
6.8	+ 8.7
7.2	+10.0
7.6	+ 8.7
8.0	+ 5.0
8.4	0
8.8	- 5.0
9.2	- 8.7
9.6	-10.0
10.0	- 8.7

Graph the displacement of the weight as a function of time. Describe the graph. Does it resemble any of the functions you studied so far? How? Or how is it different?

Investigation 2.17

A radioactive substance is one which spontaneously changes or "decays" into other substances. For example, the radioactive element radium is known to produce helium gas and another gas called radon (since it comes from radium). Radon, in turn, also is radioactive and it decays into helium and a new element, polonium. Such a chain of disintegrations would continue until only nonradioactive products (called "daughters") remain.

It is impossible to tell when any individual atom of a radioactive substance will decay. However, given a very large number of atoms, one can predict with great accuracy what they will do over a long period of time. So, while a scientist can never be certain what a particular atom will do in the next second, she can say with considerable accuracy what a billion of those atoms will do in the next million years.

Radioactivity is usually measured in a unit called the "half-life." This is the time during which half the original quantity will decay. It is, of course, a matter of probability, but since we are considering a very large number of atoms the probability model fits quite well.

In this activity you will simulate a radioactive decay using dice. Begin with a large number of dice -- 50 to 100 might be an appropriate range. Designate one face of each die as indicating decay -- for example, if a 1 comes up, that "atom" has decayed. Now, roll all the dice. Remove all those which have decayed. Roll the remaining (still radioactive) dice. Again remove all those which decay. Repeat the process until all have decayed. Record the following data in a chart:

Number of the roll	Number of dice rolled	Number decayed	Number remaining
1			
2			
3			
*			
*			

(1) Do the simulation as described above and complete the chart.

(2) For each roll, compute the theoretical probability of an atom decaying and, using the data in the chart, also compute the experimental probability for that roll. Add the experimental probabilities as another column in the table.

(3) Determine the half-life for the above. How many "atoms" remain after one half-life? After two? After three?

(4) Plot the number of radioactive atoms as a function of the number of the roll. In a different color on the same graph plot the total number decayed thus far as a function of the number of rolls.

(5) Using the data from the chart, for each roll calculate the ratio of the number of "atoms" remaining active to the number rolled.

(6) Decide how you could use the information from the experiment to estimate the "age" of the set of dice at some given time.

Now, repeat the experiment several times to determine the following:

Does the outcome of the activity depend on the size of the original pile?

What happens if you redefine a "decay" to be either a 1 or a 6?

Investigation 2.18

Another model of "decay" is found in the behavior of bouncing balls. In this experiment you are to drop a ball from a convenient height so that you can measure how high it rises on (at least) 5 or 6 bounces.

(1) Perform the experiment.

(2) Record the number of bounces, the height of the ball on each bounce, and the ratio of the height of each bounce to the height of the previous bounce.

(3) Graph height as a function of bounce number.

(4) Graph the ratio of the bounces by plotting the height of the (n+1) bounce as a function of the height of the nth bounce.

(5) Now repeat the experiment several times so that you use at least 3 different balls and at least 3 different starting heights for each ball.

(6) Compare the results from all the above and summarize your conclusions.

(7) The above experiment was also performed with a "superball." Following are the data collected. Graph these data as you did above. Does the superball obey your theory and explanation?

Bounce Number	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Height (Meters)	5	4.5	4.1	3.6	3.3	3.0	2.7	2.4	2.2	2.0	1.8	1.6	1.4	1.3	1.1	1.0

Teaching Notes

The investigations in this chapter involve the concept of a function. Key ideas to stress in discussions include: What do we mean by a variable? What do we mean when we say that a function is well-defined? What are the domain and range of a function? What is meant by a one-to-one function? Is every function one-to-one? What is an increasing function? A decreasing function? A constant function?

In all of these investigations, class discussion should focus on variation (change): What is variable? Which changes in the system depend on change elsewhere (dependent vs. independent variables)? Which aspects of the system do not change (constants)? Are there elements of the system which can change but which do not affect the outcome (irrelevant variables)? Students should note different patterns of change, but in all cases the variations can be determined and predicted, and we can describe them mathematically. Several types of variation (functions) are investigated as follows:

Investigations 2.1 through 2.5 involve linear relationships. After pupils have completed these or similar activities, they should consider the similarities and differences of the set of investigations. For these, all are linear functions ($f(x) = mx + b$); some are direct proportions ($f(x) = mx$). Thus, for example, we see that doubling the diameter of a circle doubles the circumference (Investigation 2.2); however, doubling the weight on the spring increases but does not double the total length of the spring (Investigation 2.3). Investigation 2.4 introduces both positive and negative values of the variables; Investigation 2.5 suggests the idea of the inverse of a function.

Investigations 2.6 and 2.7 have an inverse relationship (xy = constant or $y \propto \frac{1}{x}$), but students do not always recognize this from the data. They often assume the relationships are linear since, for example, they may notice quantities like p_1 and p_2 or v_1 and v_2 are in ratios of 2:1. They fail to notice, however, that in one case the ratio is 2:1 while in the other it is 1:2. Here it is important to contrast direct and inverse functions:

Direct variation $\frac{y}{x} = k$ or $y = k * x$

Inverse variation $y * x = k$ or $y = \frac{k}{x}$

where k is a constant.

Investigations 2.8 through 2.15 involve quadratic relationships, both direct ($y \propto x^2$) and inverse ($y \propto \frac{1}{x^2}$). Pupils should observe that the graphs of quadratic functions are non-linear. For a function where $y \propto x^2$, the graph of y vs. x is a parabola, while the graph of y vs. x^2 is linear. This can be related to elementary concepts of slope, derivative, area under the curve, etc. Since quadratic relationships are frequently encountered in real situations, students should learn to recognize them from the data and to explain and predict their behavior.

Also included in this set of activities are two (Investigations 2.13 and 2.14) which require the student to formulate hypotheses, to design an experiment to test these, and to gather and interpret data. Two others

(Investigations 2.11 and 2.12) provide opportunities to apply principles from other activities to new situations.

Investigation 2.16 introduces a periodic function ($y = k \sin x$), while Investigation 2.17 and 2.18 involve exponential functions ($y = ka^x$). For the periodic function, we focus on the manner in which the values of the function repeat with a certain regularity. In the exponential functions, the independent variable is in the exponent, and the functions represent growth or decay curves. Students may confuse the graphs of exponential functions with those of inverse functions ($xy = k$); discussions should contrast these by noting the different values of the functions for selected points in the domain.

As they carry out these investigations, the students should be guided to organize their data in an appropriate table. Frequently students need help deciding what to record, how to label columns in a table, or how to arrange columns. They may also need a suggestion to write data in some ordered fashion--for example, they may record circumferences and diameters of objects in the order in which these were measured whereas it could be more helpful to write the entries in increasing order of d (or C).

Similarly, students may need help to plot graphs. They should learn to identify the independent variable and to plot it on the horizontal axis, in order to be consistent with the convention for plotting functions with the horizontal axis for elements of the domain (x) and the vertical axis for elements of the range (F(x)). Frequently, pupils need guidance in selecting reasonable scales for the axes and in recognizing that it is not necessary to use the same scale for both axes. They also must learn to graph the "best" line or curve rather than to "connect the dots."

The following notes refer to particular aspects to consider in each investigation:

Investigation 2.1: This activity can be performed using carts from the physics lab, or carts can be constructed using thread spools or small wheels. Tinker toys are easy to obtain and work well. The important thing is that the cardboard strip or light flat stick rides on the wheels in order for the distance traveled by the strip to be twice the distance traveled by the cart. In analyzing the outcome of this investigation, consider first the motion of the strip if the axle of the wheel were fixed at some point. In this case, if the wheel rotates once (360°), the strip will move forward a distance equal to the circumference of the wheel.

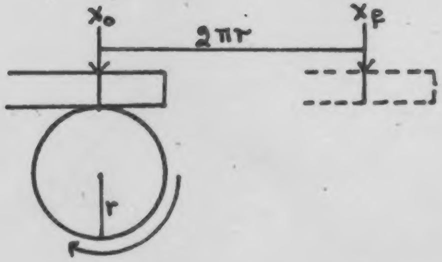


Figure 2.10

Wheel is fixed with center at x_o. If wheel rotates 360°, strip advances through distance equal to the circumference of the wheel.

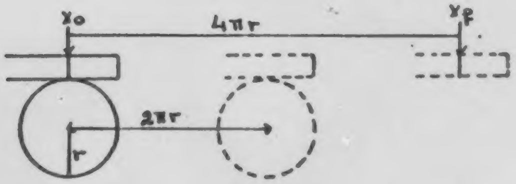


Figure 2.11

Wheel moves forward as it rotates. Wheel moves through distance of $2\pi r$ in one rotation; strip moves through distance of $4\pi r$.

However, if the wheel itself advances forward (i.e., rolls) as it rotates, then the strip travels a distance of $4\pi r$ due to the motion of the rotating wheel (as in Figure 2.10) plus the forward movement of the wheel.

Investigation 2.2: The objects used should have a wide range of diameters. Small objects, like buttons, will yield data that do not seem to fit the line as well as do the points obtained from larger objects. This can motivate discussion about accuracy of measurements and relative error. The activity also presents some practical problems of measurement: How can you be sure you have the diameter? How can you measure a round object? (Students should come up with two approaches: wrapping a string or tape around the object and rolling the object along a straight line.)

After the students record d and c in a table, they can be directed to add more columns and record (c + d), (c - d), (c * d), and (c/d). This makes it more apparent that c/d is a constant regardless of the values of c and d. Students should perform these calculations with a calculator, so it is important to discuss significant figures for c/d. This in turn motivates class discussion of experimental accuracy and precision. Finally, the graph is studied to determine its slope (approximately 3.1) which, of course, we decide to name π.

Investigation 2.3: A reasonable substitute for springs can be had with rubber bands of different length and thickness. For lighter bands, large paper clips or light washers can be used for weights.

The investigation directs the students to measure the length of the entire spring (L) with various weights (W) suspended. Thus when W=0 (i.e., no weights added), the spring has an initial length L_o. In this case, the linear graph intersects the vertical axis at L_o. If we repeat the experiment and measure only the increase in length (ΔL) as weights are added, the result is $\Delta L=0$ for W=0. Hence, the linear graph passes through the origin. These two situations can lead to a comparison of the slope of the graphs, the significance of the y-intercept, and the distinction between a linear function [f(x) = mx + b] and a direct variation [f(x) = mx].

For further discussion, consider the statement of Hooke's law: $F = -kx$. This is a description of the force exerted by the spring when it is stretched a distance, x. What is the physical meaning of the negative sign in this expression? How is this expressed in the graph of the function? How does this situation compare with the one in Investigation 2.3?

Investigation 2.4: This investigation is similar to the previous one. Here, however, forces are applied both upward and downward, and to differentiate between the two it is reasonable to assign opposite signs: positive values for lifting forces and upward deflections, negative values for the opposites. Consequently, the graph must be extended beyond the first quadrant. Caution: the pulley must be positioned so that the lifting force is vertical. If not, the force vector will have both horizontal and vertical components, as shown, and only $\vec{V}$ will be a lifting force. (This situation can suggest other more involved problems for advanced students.)

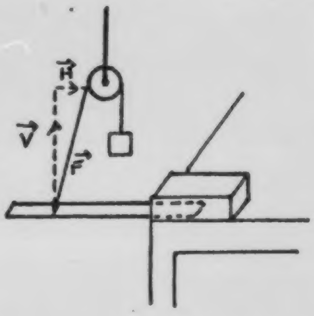


Figure 2.12

In both Investigations 2.3 and 2.4, students should repeat the activity with different springs and different blades to observe that any one spring or blade produces a linear relation between force (weight) and displacement, while from spring to spring or blade to blade there are differences which show mathematically as differences in the slopes of the graphs.

Investigation 2.5: This is a good example of an easy simulation of an activity for which the actual equipment probably is unavailable. It also reinforces concepts of temperature, degrees, and Fahrenheit and Centigrade scales. In this case, neither of the variables is necessarily "dependent" or "independent," so it is reasonable to represent either one on the horizontal axis. In all likelihood, some students will graph C on the horizontal axis; others will graph F. (If they do not, they should be directed to do so.) This will produce two graphs as follows:

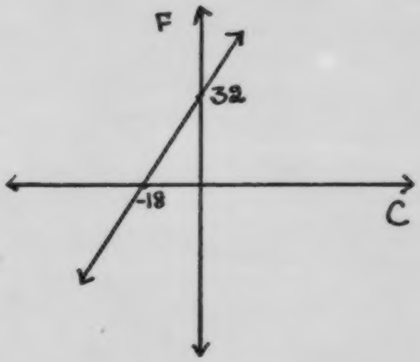


Figure 2.13

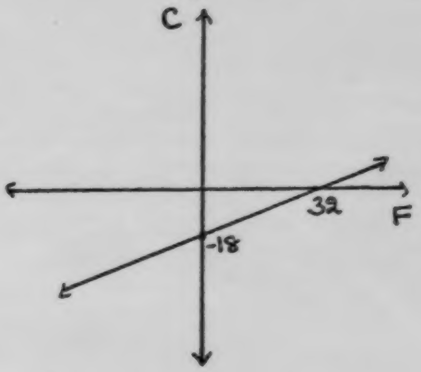
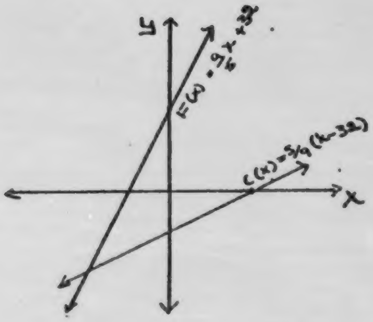


Figure 2.14

From the graphs we can derive the familiar relationships

$$F = \frac{9}{5} C + 32 \quad \text{and} \quad C = \frac{5}{9} (f - 32).$$

Further, if we graph both functions on the same coordinate system, we can introduce the concept of inverse functions and we can study the relationship between the graphs of a function and its inverse.



Investigation 2.6: If possible, an experiment or demonstration should be conducted to generate real data. If this is not feasible, hypothetical data like those given can be used. Note that this investigation involves only Boyle's Law: $p_1v_1 = p_2v_2$ or $pv = k$ at constant temperature. A logical extension is to investigate Charles' Law: $\frac{v_1}{v_2} = \frac{T_1}{T_2}$ or $\frac{T}{V} = k$ at constant pressure. The latter will again yield a direct variation similar to those found in previous investigations. Later the two laws can be combined to yield the more general gas law: $\frac{P_1V_1}{T_1}=\frac{P_2V_2}{T_2}$ or $PV = kT$. Note that these laws are for temperature expressed on the absolute or Kelvin scale ($^oK = {}^oC + 273$). Note also that they describe ideal gases. Real gases will deviate from these laws especially at high pressure and low temperature extremes. However, we will not encounter these situations in the school classroom.

Investigation 2.7: The situation given here is only illustrative of many which could be studied. The relationship of speed to time in this investigation leads to other questions about distance, velocity, and acceleration in the investigations which follow.

Investigation 2.8: Here the student is introduced to uniformly accelerated motion. (Actual experimental data can be substituted for the hypothetical values given in the investigation.) The graph of distance vs. time (a quadratic function) is contrasted with the graph of speed vs. time (a linear function). Note that here we speak of distance and speed rather than of displacement and velocity, since we are not immediately concerned with the vector nature of these quantities. Later extensions can consider this characteristic thus yielding graphs in which the parabola is concave downward (as in the graph of distance vs. time for a ball thrown vertically upward).

In discussing the investigation, relate the average speed in an interval to the slope of the line segment joining the endpoints of the interval. As the interval narrows, the segment approaches the tangent, and the slope of the tangent represents the instantaneous speed at the point of tangency.

Mathematically,

$$\text{slope} = \frac{y_2-y_1}{x_2-x_1} = \frac{\Delta y}{\Delta x} = \frac{\text{distance traveled}}{\text{time}} = \text{average speed}$$

and in the limit, as $\Delta x \rightarrow 0$,

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

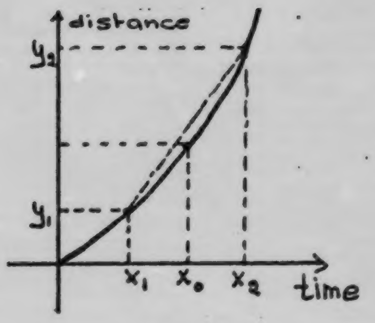


Figure 2.15

Hence, we have an application of the derivative, and, for advanced students we can extend the discussion to elementary concepts of calculus.

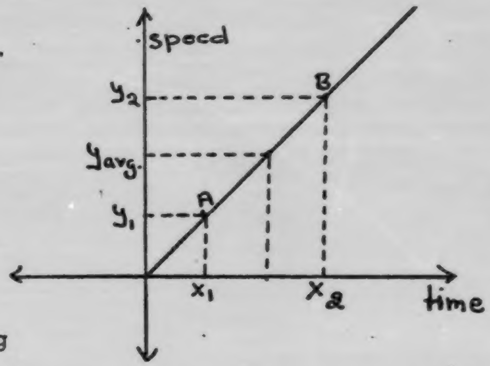


Figure 2.16

From the graph of speed vs. time, we can find the average speed in an interval:

$$\text{Average speed during interval } (x_2 - x_1) = \frac{y_2 + y_1}{2}$$

And the distance traveled during this interval is given by:

$$\text{distance} = (\text{average speed})(\text{time}) = y_{avg}\ (x_2 - x_1)$$

But this is equivalent to the area of trapezoid ABx_2x_1 in Figure 2.16, or the area under the curve from x_1 to x_2. Again, we can extend the discussion to concepts of calculus if appropriate for the given students.

The two problems posed at the end of the investigation warrant attention. In the first case, students tend to oversimplify by suggesting that they would maintain a steady speed and measure the time, then calculate d = speed * time.

However, it is unrealistic to assume one could drive across the desert at a uniform speed. A better suggestion is to record the speed (rate) at fixed intervals. From the graph of speed vs. time we can approximate the distance by finding the area under the curve. In the second problem, students frequently average the two rates. This, however, ignores the fact that the woman must drive twice as long at the slower speed. Hence, the times to go and to return are given by $t_1 = \frac{d}{30}$ and $t_2 = \frac{d}{60}$ and the average speed is

$$\text{average speed} = \frac{2d}{t_1 + t_2}$$

which is 40 mph, not 45.

Investigation 2.9: The concepts involved here are the same as in the previous activity, but we now have the added elements of free fall and the acceleration due to gravity--concepts which we use again in Investigation 2.11. Extensions of this activity could include problems of gravitational acceleration on other planets. This allows us to note that the acceleration is uniform at the surface of each planet, but that the constant values of A_g varies from one planet to another, both because of differences in the masses of the planets and because of differences in their sizes. Or we can extend to problems of gravitational acceleration for bodies at different distances from the center of the earth to examine the relationship that $A_g \propto \frac{1}{d}2$, or to bodies of different masses at a fixed distance from the center of the earth to note that $A_g \propto m$.

The following values of A_g may be used in such problems:

Body	Surface Acceleration ft/sec^2	m/sec^2
Mercury	11.5	3.53
Venus	27.8	8.53
Earth	32	9.80
Mars	12.2	3.72
Jupiter	84.5	25.87
Saturn	37.4	11.47
Uranus	29.4	9.02
Neptune	44.8 (?)	13.72 (?)
Pluto	?	?
Moon	5.4	1.67
Sun	896	274.40

Investigation 2.10: This investigation shows a case where acceleration is not uniform--hence the speed vs. time graph is not linear. It is important that students encounter examples such as this so they do not develop the false conclusion that the speed/time graph will always be a straight line if the distance/time graph is non-linear. They likewise should be asked to graph speed vs. time for cases of constant speed (hence acceleration is zero and the speed vs. time graph is a horizontal line.)

Investigations 2.11 and 2.12: These activities apply concepts from the previous investigations. Students generally enjoy these and consider them "fun." A nice way to relate the two is to make a motion book for the case of the ball rolling off the table.

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Investigations 2.13 and 2.14: These are purposely stated briefly and with no specified procedures because these activities are especially good for teaching concepts of experimental investigation. Students should hypothesize variables which might affect the outcome: length, mass of object on the pendulum or spring, initial displacement, material of spring or pendulum, etc. Then they must design experimental procedures for systematically changing one variable while controlling others. The investigations also are recommended because they frequently force students to confront unexpected results. For example, it is not uncommon for students to be so convinced that the mass of a pendulum determines the period, that they will try to force the data to support their hypothesis despite experimental evidence to the contrary.

Both investigations yield quadratic relationships. For the pendulum, a good approximation is given by

$$T = 2\pi \sqrt{\frac{l}{g}}$$

where T is the period, 1 is the length of the pendulum and g is the gravitational acceleration. Hence T depends only on 1. Do not expect students to determine the above equation, however. Students almost always reach the qualitative description that "the longer pendulum has a longer period." To encourage them to quantify their conclusions, ask questions like, "What should I do to the length in order to double the period?" or "If one pendulum is 25 times as long as another, how do their periods compare?" For most purposes it is sufficient that students recognize the relationship $T \propto \sqrt{l}$ or $T^2 \propto l$.

In the case of the oscillating spring,

$$T = 2\pi \sqrt{\frac{m}{k}}$$

where T is the period, m is the mass (inertia) of the object on the spring, and k is a force constant which depends on the stiffness of the spring. Once again, it is sufficient to expect students to determine that $T \propto \sqrt{m}$ or $T^2 \propto m$. There are, however, two other points worth noting: First, unlike the pendulum, here the period does depend on the mass. Second, we can contrast the oscillating spring with the spring in Investigation 2.3. (If possible, use the same springs for both activities.) In the first investigation (2.3), we found a linear function, while in this one we find a quadratic function.

Extensions of these investigations can take students into a study of simple harmonic motion and of the relationship between uniform circular motion and simple harmonic motion. In particular, as a point moves with uniform speed around the unit circle, the projection of that point on the axis defines a simple harmonic motion which is defined by a sinusoidal function.

Investigation 2.15: This investigation is representative of the many inverse square laws which describe natural phenomena. Here we use Coulomb's Law: $F = k \frac{q_1 q_2}{r^2}$ (similar investigations could include gravitational or magnetic forces). Since we are trying to describe the relationship between distance and force, it is convenient to convert the given values to ratios: let $r = 5.3(10^{-11})$m and express other distances as multiples of r. Similarly, let $f = 8.1(10^{-8})$nt and express the other forces as multiples of f. Thus, for example, at a distance 2r we have a force of $\frac{1}{4}$ f.

Investigation 2.16: For the third time we have an investigation involving springs. This time the function is a periodic function [$y = 10 \sin (wt + \frac{3\pi}{2})$ where, for the example given, $w = \frac{5\pi}{12}$]. The important concept is the periodic nature of the function--i.e., the fact that y repeats every 4.8 sec. Hence, $f(t) = f(t + 4.8)$ for all t. The function could be continued, but the data given are sufficient to exhibit two complete periods of the function. For the one chosen here, note that the situation is idealized, since the motion of a real spring would be damped and the amplitude of the displacement would decrease. However, this simplified situation better illustrates the point we wish to emphasize.

Investigation 2.17: Here again we simulate a real situation. However, teachers with access to counters and isotopes of reasonably short half-life might prefer to study real data. The investigation can lead to discussions of probability and the manner in which the model approximates actual events. Encourage students to try the activity with larger sets of dice to see how the number of dice affects the outcome. If one has access to a sufficient quantity of other regular solids such as tetrahedra or dodecahedra, the experiment should also be performed with these to contrast the theoretical and experimental probabilities for solids with different numbers of faces.

Investigation 2.18: This activity also involves an exponential function, but here students make their own measurements of the real event as opposed to the simulation in the previous investigation. Several practice runs will be necessary for students to develop skill at measuring the ball's heights. A meter stick taped to a wall is helpful. The activity seems to work best with three students: one to drop the ball, one to measure and call out the heights of the bounces, and one to record the data.

Other Applications

The following are some of the many applications of functions in science. Each could be developed into classroom investigations.

Linear Functions

1. The rate at which a cricket chirps is affected by temperature. Hence, by counting the number of chirps per minute (n) we can approximate the temperature (t) in degrees Fahrenheit from the function $t = \frac{n}{4} + 40$.
2. In an electrolysis of water, the volume of hydrogen or oxygen liberated is a function of the electrical charge which flows through the system. (Alternatively, for a constant current, the volume of the gas is a function of time.) Also, the ratio of hydrogen to oxygen is a constant.
3. The acceleration of a body of fixed mass is proportionate to the force applied ($F = ma$).
4. For a given material, the mass of a sample is proportional to its volume (Density $= \frac{m}{v}$).
5. The length of a shadow is dependent on the height of the object (the angle of incidence of the light being held constant).
6. When a metal rod is heated, the increase in its length is a function of the change in temperature. The rate of expansion depends on the metal ($\Delta l = l_o \bullet \Delta t \bullet k$ where l_o is the initial length and k is the coefficient of linear expansion for the metal).
7. When a liquid is heated, the increase in its volume is a function of the change in temperature. The rate of

expansion depends on the liquid ($\Delta v = v_e \bullet \Delta t \bullet k$ where k is the coefficient of volume expansion for the liquid).

8. In a tank of water, the pressure of the water depends on the depth ($P = h \bullet D$ where h is the depth [height] of the liquid and D is its density).
9. The ratios of elements in various compounds are fixed (law of constant proportions). Hence, the amount of each element present is a function of the total sample size.
10. The velocity of sound in air is 1090 ft/sec at 0°C, and the speed increases 2 ft/sec for each degree Centigrade rise in temperature. (Extension: Since the velocity of light is $186(10^3)$ miles/sec, we can define a function to determine the distance of a lightning flash.)
11. A substance such as moth balls is heated enough to melt the material. The substance then is allowed to cool to room temperature (or below), and the temperature is recorded at fixed intervals. The resulting graph of temperature vs. time shows a decreasing linear function until the substance begins to solidify; then a constant function ("the plateau") until freezing is complete; and, finally, another decreasing linear function until room temperature is achieved.
12. The time required by light from distant stars to reach the earth is a function of distance.

13. The amount of heat required to raise the temperature of a substance a fixed number of degrees is proportional to the mass of the substance.

14. In the respiration of plants and animals, the amount of O_2 used is proportionate to the amount of CO_2 produced ($C_6H_{12}O_6 + 6O_2 \rightarrow 6CO_2 + 6H_2O$ + energy). (Conversely, in the photosynthesis of green plants, $6CO_2 + 6H_2O$ + energy $\rightarrow C_6H_{12}O_6 + 6O_2$.)

Inverse Functions

1. For a balance beam with a fixed force acting on one arm, the mass needed to restore balance is inversely proportional to its distance from the fulcrum.

2. The acceleration produced by a given force is inversely proportionate to the mass of the object being accelerated ($\frac{F}{m} = a$).

3. The force needed to roll a heavy object up an inclined plane from the ground to a specified height is inversely proportional to the length of the incline (neglecting friction).

4. The frequency of a vibrating violin string varies inversely as the length of the string (provided other conditions are constant).

Quadratic Functions

1. The intensity of illumination from the light source is inversely proportional to the square of the distance from the source.
2. The area of a picture projected on a screen varies directly with the square of the distance from projector to screen.
3. The weight of an object varies inversely with the square of its distance from the center of the earth.
4. The rate of evaporation of water from a cylindrical container at fixed temperature and atmospheric conditions is proportional to the exposed surface of the water--hence, to the square of the radius of the cylinder.
5. The frequency of a vibrating violin string varies directly as the square root of the tension (provided other conditions are constant).
6. The frequency of a vibrating violin string varies inversely as the square root of its weight per unit length (other conditions being constant).
7. At constant temperature and pressure, the rates of diffusion of gases are inversely proportional to the square root of their molecular weight (Graham's Law: $\frac{R_1}{R_2} = \sqrt{\frac{m_2}{m_1}}$ or $R \propto \frac{1}{\sqrt{m}}$ Hence, $m \propto \frac{1}{R^2}$).

8. The kinetic energy of a gas molecule is given by $KE = \frac{1}{2} mv^2$. Also, the average kinetic energy of molecules is constant at a given temperature. Thus, for fixed temperature, $v^2 \propto \frac{1}{m}$. (For example, a hydrogen molecule will have four times the velocity of an oxygen molecule at the same temperature.)

Periodic Functions

1. A stretched string vibrates with a constant frequency. A particular point on the string, P, is studied to determine the displacement of P from its rest position. That displacement as a function of time is a sinusoidal function.
2. Alternating current ("AC") continually changes in magnitude and periodically reverses direction--i.e., it increases from zero to a maximum in one direction through the wire, decreases to zero, and then increases to the maximum in the other direction before decreasing to zero again. The graph of the instantaneous current (or voltage) as a function of time is a periodic function.
3. A unit vector rotates counterclockwise with a constant angular speed. The projection of the vector on the horizontal axis is a sinusoidal function of time.

Exponential Functions

1. The growth of bacteria which double every x minutes is an exponential function of time. (Number of cells $\propto 2^{\frac{t}{x}}$ where t is the time elapsed and x is the number of minutes per generation.)
2. Carbon-14 decays with a half-life of 5700 years. (This permits the C^{14} method of dating fossil remains of living organisms.)
3. The amount of light which passes through a pane of glass depends on the thickness of the glass.
   (a) Suppose a certain type of glass allows 4/5 of the light to pass through a standard unit of thickness. How much light could pass through 2, 3, 4, 5 ... thicknesses of this glass?
   (b) Suppose 1 cm of glass transmits 81 percent of the light reaching it. How much light would pass through .5 cm? .2 cm? .1 cm?
4. Suppose a photographic enlarger is capable of producing an enlargement up to 2.5 times the size of the original. At maximum setting of the enlarger, how large would the pictures be if the enlarger is used once? twice? three times? five times? etc.
5. Water blocks out light, and the amount of light transmitted depends on the purity of the water, the plant and animal life, etc. Suppose a one meter depth of

water passes 85 percent of the light. Determine the amount of light at depths down to 12 meters.

6. The diaphragm of a camera lens is marked in numbers called "stops" (11, 8, 5.6, 4, 2.8 are common stops). From any stop to the next smaller number, the amount of light reaching the film at a given shutter speed doubles.
   (a) Graph the above relationship.
   (b) If the 2.8 stop admits L light, how much light is admitted at the other stops for the same shutter speed?
   (c) If 5.6 admits L light, how much for the other stops at the same speed?
   (d) If proper exposure for a particular picture is obtained at 11 with a shutter speed of 1/25 sec, what conditions would give the same exposure at each of the other stops?

7. A man used a water soluble paint. When he finished painting, 4 fl. oz. of paint remained in his brush. He dipped the brush into a quart (32 fl. oz.) of clean water and mixed the solution thoroughly. When he was finished, the brush still contained 4 fl. oz. of liquid, now a solution of paint and water. He repeated the cleaning process with a quart of clean water. After each washing, 4 fl. oz. of liquid remained in

the brush. How much paint is in the brush after each of the first 6 washings? Will the brush ever be completely clean? How much paint is left after n washings?

8. The frequency of a musical tone doubles with each octave. The frequency of middle-C is 256 cycles per second. Plot the frequencies of the "C-tone" for three octaves above and below middle-C.
9. For many chemical reactions, a rise in temperature of $10^{o}C$ approximately doubles the reaction rate.

Other Functions

1. The relationship between period of revolution (P) of a planet around the sun and its distance (D) from the sun is given by Kepler's third law: $P^2 = D^3$ (P is measured in years, D in astronomical units).
2. The volume of a sphere increases with the cube of the radius ($V \propto r^3$).
3. The gravitational attraction between two bodies varies directly as the masses of the objects and inversely as the square of the distance between them (Joint variation: $F = G \frac{m_1 m_2}{r^2}$).
4. The electrical force (attraction or repulsion) between two charges varies directly with the strength of the charges and inversely with their separation ($F = k\frac{q_1 q_2}{r^2}$).

5. The amount of current (I) flowing in a wire varies directly with the voltage (V) and inversely with the resistance of the wire (V = IR).

6. The magnetic attraction or repulsion between two magnets varies directly with the strength of the magnets and inversely as the distance between them ($F = k \frac{M_1 M_2}{r^2}$).

Chapter 3

Measurement

Measurement is an important process both in science and in mathematics. Essential to measurement are two processes: a comparison is made between the quantity to be measured and some standard or nonstandard unit, and a numerical value is assigned to describe that comparison. The types of measurement we most frequently encounter are distance (linear), area (surface), volume (capacity), direction, mass, temperature, and time. Learning tasks include making direct and indirect comparisons between objects, selecting appropriate units for a measurement task, ordering, estimating, representing measurements quantitatively and graphically, scaling, measuring inaccessible objects, and evaluating the accuracy of a measurement.

Some of our everyday measurement tasks can be accomplished by direct measurement in which the measuring instrument is applied directly to the object to be measured. This is the case when we lay a ruler next to a board, stretch a tape around the waist, step on a scale, or immerse a thermometer in a liquid. Many other times, such direct measurement is either impossible or inconvenient and we must rely on indirect approaches. In this case, we make direct measurements of one or more related quantities and use these to compute or approximate the desired measurement.

Indirect measurements like these may be necessary because the quantity of interest is too big or too small or too fast or too distant or otherwise inaccessible. At other times, indirect measurements provide convenient shortcuts as, for example, in measuring the area of a rectangle. Here, it may be possible to cover the rectangle with unit squares, but it is more convenient to make linear measurements of length and width and to compute area from these.

Students should engage in direct measurement activities to develop fundamental concepts and skills and to learn appropriate units of measurement. Such activities are included in most science and mathematics textbooks and are not considered here. The investigations in this chapter involve indirect measurements which require more complicated applications of measurement techniques.

Investigations

Investigation 3.1

One of the measurement problems we must face involves measuring very small quantities. Devise a method and measure each of the following:

(a) The thickness of a piece of tissue paper

(b) The volume of one raindrop

(c) The mass of one crystal of table salt

(d) The area of the head of a straight pin

(e) The diameter of a strand of sewing thread or a strand of your own hair

How accurate do you think your measurements are? What are the most likely causes of error in the measurement? What could you do to improve your results?

Investigation 3.2[a]

We observed in our motion experiment that falling objects accelerate under the constant force of gravity. Experiments have shown that on the surface of the earth, the acceleration due to gravity is approximately 980 cm/sec^2. We will use this information to get an estimate of human reaction time -- the time during which your eye observes something, a message is sent to the brain, a decision is made, and appropriate muscles respond.

Experiment

You will need a partner for this experiment. The person being tested sits with hand outstretched and thumb and forefinger ready to close and catch a falling object. The partner holds a 30 cm ruler with the zero mark between the catcher's open fingers. The person holding the ruler drops it whenever he or she chooses. The other catches the falling ruler and notes the mark where the catch is made.

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[a] For more complete discussions and related problems, see The Man-Made World, New York: McGraw-Hill, 1971.

Assuming that the dropper simply lets go of the ruler (i.e., does not thrust it downward), the distance the ruler falls can be used to estimate the reaction time of the person catching. Repeat the experiment many times for each partner to get a better approximation of each one's reaction time. How many trials do you think you should make? Do you think that "practice makes perfect" in this case? (Note: Try this experiment again sometime when you are very tired or exhausted. Does it make a difference?)

Application:

Suppose you are driving your car and have to stop suddenly. How long will it take to stop and how far will the car go in that time? The following sequence of events must take place:

(a) The eye observes a danger (yellow light, oncoming car, child running into the road, etc.) A message is sent to the brain, a decision is made, a message is sent back to the right foot, the foot begins to move.

(b) Time is needed for the foot to move to the brake and to push down the brake pedal.

(c) After the brake is applied, the car decelerates or slows down. (The greater the deceleration the more the passengers are thrown forward and the more dangerous and uncomfortable the stop.)

Despite the limitations of the experiment performed above, results have shown that minimum human reaction time is about 0.2 seconds. (What was yours?) In the driving situation the reaction is more complex than just closing the fingers, which are already in position for the expected fall. For the driving case, actual reaction time seems to be about twice as long as that measured in the experiment. Calculate how far a car would go if you were driving it at, say, 10, 20, 30, 40, 50, or 60 mph. (Use your own reaction time and also the minimum time mentioned above.)

Once your foot starts to move, there is still the time needed for the foot to hit the brake. Again, experiments with typical human beings have shown that this time also was about equal to the reaction time -- and that was under conditions when subjects were waiting for a certain signal. In reality, the usual time to hit the brake under actual driving conditions is also about twice the reaction time. Thus, the total time for reaction to the warning and hitting the brake is about four times the reaction time from the experiment. Again, calculate how far the car would travel for you at each of the speeds listed above. How far under the minimum reaction conditions? Finally, once the brake has been depressed, the car still must decelerate. The following minimum stopping distances are based on tests made by the Bureau of Public Roads. Reaction time is taken as 0.75 sec.

Miles per hour	Reaction distance (feet)	Braking distance (feet)	Total stopping distance (feet)
5	5.5	4	9.5
10	11	8	19
15	16.5	13	29.5
20	22	20	42
25	27.5	28	55.5
30	33	40	73
35	38.5	52	90.5
40	44	72	116
45	49.5	92	141.5
50	55	118	173
55	60.5	148	208.5
60	66	182	248
65	71.5	220	291.5
70	77	266	343

Plot the above data on a graph using the same set of axes to graph both reaction distance and total stopping distance. How is braking distance shown on the graph?

Prepare a table and graph similar to the above but using your own reaction time to calculate reaction distance.

Investigation 3.3

One of the most famous experiments in the history of science was performed by the Greek geographer Eratosthenes in the third century B.C. He is credited with having measured the circumference of the earth. For the experiment, Eratosthenes selected two points in Egypt: Alexandria, on the Mediterranean Sea, and Syene, near the present location of the Aswan Dam. The two locations are approximately on the same meridian. At the time of the summer solstice, the noon sun shown directly down into a deep well at Syene.

(Hence, a pole stuck in the ground at Syene cast no shadow at noon.) However, at the same time, a pole stuck in the ground at Alexandria cast a measurable shadow. Figure 3.1 represents this situation.

The experiment is based on the following assumptions:

(1) The earth is a sphere

(2) A plumb line from any point on earth points toward the center of the earth

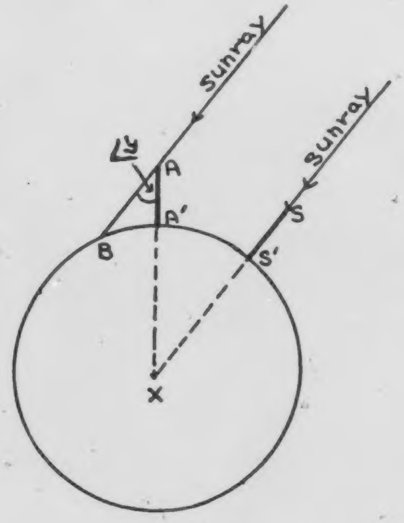


Figure 3.1

(3) The sun and stars are so far away from earth that sunlight reaching two different points on earth travels in lines that are essentially parallel.

In Figure 3.1, AA' and SS' are the poles at Alexandria and Syene, respectively. A'B is the shadow at Alexandria caused by the sunray AB. Since AA' and SS' are erected vertically, the extensions of these lines meet in the center of the earth. Further, since the rays at the two places are assumed to be parallel, the angle x at the center of the earth is equal to the angle y between the pole and the sunray at Alexandria. By measuring the length of the shadow and the height of the pole, one can calculate the size of angle y, and hence of angle x. But the angle at the center of the earth (x) is equal in degrees to the arc from Alexandria to Syene, a measurable distance. (Known by Eratosthenes to be 5000 stades, a unit which historians believe to be about 0.1 mile.)

In his measurements, Eratosthenes found the angle y (and hence x) to be 7.2 degrees. Since 7.2/360 is approximately 1/50 of the circumference, he could then approximate the circumference of the earth -- and from that also the diameter. Complete the calculations and compare the results with the currently accepted value of 24889 miles from the circumference around the equator.

A. Review the logic in the above analysis. Can you justify each of the steps in the process?

B. Examine the three assumptions noted. How does each of these affect the problem?

C. Are there any other assumptions implied in the above (besides the three listed)? If so, what are they and how do they affect the problem?

On the next page is a description of a classroom simulation of Eratosthenes' methods. Complete the simulation for several different circles and check the accuracy of your measurements against the actual sizes.

Measuring circumference and diameter -- Eratosthenes' method

(1) Near the edge of a sheet of paper draw the arc of a large circle such as the one shown at the left. (Better yet, have a partner draw the arc for you, keeping a record of the diameter used but not telling you that diameter. After you have completed the activity, have your partner check the accuracy of your results).

(2) Pick two points on the arc fairly far apart but not too close to the ends of the arc. Call these A and S.

(3) Construct perpendiculars at A and S. (Recall from geometry what is meant by a perpendicular to a circle.) These represent the vertical poles at the two points.

(4) Obviously, if you extend the two verticals they will meet at the center of the circle. However, Eratosthenes could not do that on the earth, so you may not do it either. What he could do was compare the directions of the two verticals with a standard direction which he took to be the direction to the sun. You may do the same in the following manner:

(5) Take your paper to a sunny place and position it so that a pencil held perpendicular to the paper at the right side -- i.e., here.............................. will cast a shadow across the page and along the Syene vertical. (To do this you may have to hold the page almost edge-on to the sun.)

(6) Now, holding the page in exactly the same position (you will no doubt need help), move the pencil along the edge of the page until its shadow falls on A (Alexandria). Mark the location of the pencil point when this occurs. When you put the page back down, draw the line from the pencil point position to Alexandria. (Why?)

(7) Now apply the techniques described on the previous page to complete the experiment. Explain what you are doing and why as you go through the procedure. Compare your results with the answer held in trust by your partner. Compute your percent of error. If your results are off, what are the likely causes of this? Suppose you plan to do this activity, but the sun does not shine. How could you modify the procedures so that you could complete the activity as scheduled?

Investigation 3.4

When rock masses within the earth move along a fault, energy is released which travels as a disturbance (wave) which we call an earthquake. Because earthquakes send out different kinds of waves, we are able to locate the center of the disturbance. (We call the place where the fault occurs the focus; the spot on the surface of the earth directly above the focus is the epicenter.)

There are three principal types of earthquake waves:

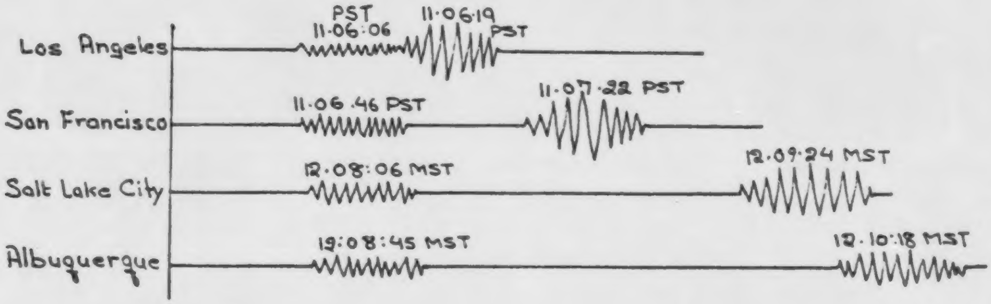
p-waves: Are the primary waves. These are pressure waves which are longitudinal--i.e., energy is traveling in a push-pull manner along the direction of the wave. P-waves travel through solids, liquids, and gases and are faster than the other kinds.

s-waves: Are the secondary waves. They are shear or transverse waves and the energy is transmitted perpendicular to the direction of the wave. S-waves are transmitted only through solids and they travel at a slower rate than p-waves.

l-waves: Are the so-called "long waves" that are created when the p and s-waves reach the earth's surface at the epicenter. L-waves travel only along the surface of the earth, and they are slower than either p- or s-waves.

When a seismic disturbance occurs, all three types of waves are created. They do not travel at the same speed, however, so if a seismograph located at some point X records the disturbances which reach it, the p-waves will arrive first, followed by the s-waves and finally the l-waves. The further X is from the epicenter of the quake, the greater will be the time delay between the p and s-waves. Hence, by knowing the speed of p- and s-waves in the region of the seismograph station, an observer can record the time lag between the first arrival of p- and s-waves, and calculate the distance to the quake.

But this information is only sufficient to determine the distance of the quake from X. Thus, the epicenter may be located anywhere on a circle of radius d_x around X. However, if two other stations, Y and Z, also record their distances from the disturbance, then the intersection of the three circles plotted from these data will locate the epicenter.


Los Angeles
PST
11·06:06
11·06·19
PST
San Francisco
11·06·46 PST
11·07·22 PST
Salt Lake City
12·08:06 MST
12·09·24 MST
Albuquerque
12:08:45 MST
12·10:18 MST

Above are seismograph records for four cities. Which city is closest to the center of the quake? If the average velocities of p- and s-waves in the region are 3.80 mi/sec and 2.54 mi/sec, respectively, how far is the epicenter from each station? Locate it on a map. Approximately what time did the quake occur?

Investigation 3.5

Measuring distances in space is, of course, a difficult problem since we cannot pace off a length or stretch a measuring tape. Astronomers rely heavily on a principle called parallax to make such measurements. Parallax is easily demonstrated as follows:

> Hold your arm outstretched in front of you with the thumb pointing upward. Look at your thumb with one eye closed. Now look at it with first one eye then the other. Note how the thumb appears to change position against the background of some distant object such as a tree across the street. Repeat the demonstration a couple more times with the thumb at different distances from you. Note how the apparent side-to-side motion of the thumb changes at different distances from you.

Parallax is based on this apparent motion of a near object against a background of distant objects due to the motion of the observer. In the case of the thumb (near object) against a distant background (trees), the "observer's motion" is caused in your head by the separation between your eyes. In astronomy, the near object might be the moon and the distant objects the stars. The observations might be made by astronomers at two different observatories. Consider the following:

At the same instant, two astronomers observe the moon -- one from point A where the moon appears at its high point on the sky, and one from point B where the moon appears on the horizon (see Figure 3.2).

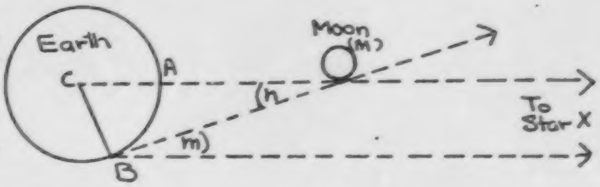


Figure 3.2

Now suppose both observe a particular star. Since the stars are so far away from earth, the lines of sight from two points on earth to a star are essentially parallel. So suppose that some star appears on a line with one edge of the moon as viewed from point A. Then, due to the parallax principle, observer B will see the same edge of the moon in a different position relative to star X. Thus, the observer at B can measure the angle of parallax (angle m) between the moon and the star. This angle also is equal to angle n in the diagram.

Now consider triangle BCM. Since the angle of parallax is very small (less than one degree), the triangle BCM is very nearly an isosceles triangle. The known radius of the earth (BC) is approximately 3960 miles. You now have enough information to calculate the distance to the moon. (Note: The parallax angle in this case is about 57 minutes.)

Study the principles involved in parallax measurement. Then complete the next activity in which you use parallax to measure the distance to an object.

1. An alidade is an instrument consisting of a rule equipped with sights which can be used to determine directions.

   You can make a simple alidade by taping a cardboard sight to each end of a meter stick. A pattern is shown in Figure 3.3 It should be enlarged to approximately 6 inches finished height. The sights should be taped to the meter stick so that the triangular piece and the front strip are vertical when the meter stick is laying on the table. The finished instrument is shown in Figure 3.4a. Using a pin or compass, make a small hole in the rear sight. The hole should be just large enough to allow you to see the front sight.

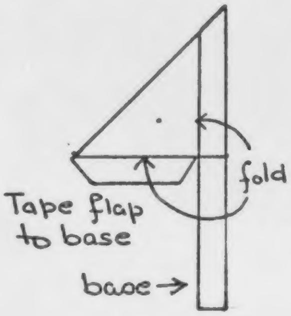


Figure 3.3

   An alternative method is to tape or glue a drinking straw parallel to the edge of a ruler as in

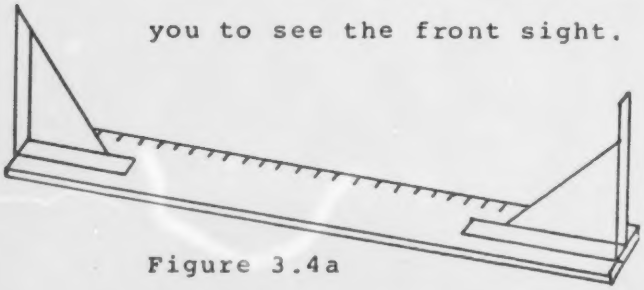


Figure 3.4a

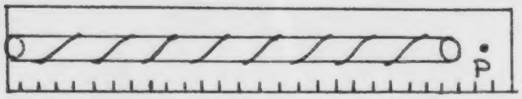


Figure 3.4b

Figure 3.4b. A pin is inserted at one end (P in Figure 3.4b) to serve as a hairline for sighting.

To use this alidade, sight through the straw from the end away from the pin and align the pin with the object being sighted.

2. Place your paper on a horizontal surface such as a small table or stool. (It is a good idea to tape the paper to a thick cardboard.) Insert a straight pin vertically into the paper near the center of the page but close to the edge away from you (see A in Figure 3.5). Now place your meter stick against the pin A and sight on some very distant object such as a tall building, smoke stack, water tower, or steeple several blocks away (see Y in Figure 3.5). Taking care not to move the meter stick after you sight on the object, draw a line along the edge of the meter stick. This line will give your reference direction.

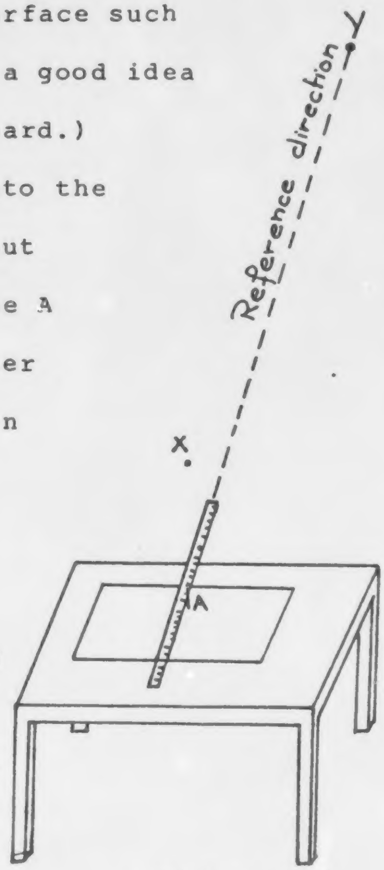


Figure 3.5

3. Now, without moving either your paper or pin A, rotate the eyepiece end of the meter stick until you sight on

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the near object whose distance you want to measure (see X in Figure 3.5). Be sure to keep the meter stick against the pin A. When you have found X in your sights, draw another line along the edge of the meter stick. Call this line l_1. Your paper now should resemble Figure 3.6.

4. Before you move the paper, mark the point B from which you sighted on X. You can do this by placing a stick in the ground or a small vertical rod in a stand or any other convenient marker which can be seen from a distance.

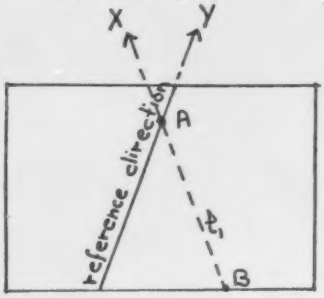
X
Y
A
reference direction
l_1
B

Figure 3.6

5. Move your table or stool to a new location about 4 or 5 meters from your first sighting position. The direction from the first sighting (B in Figure 3.7) to the second (C) should be approximately perpendicular to the direction from B to X. Be sure that you can see both X and Y from position C.

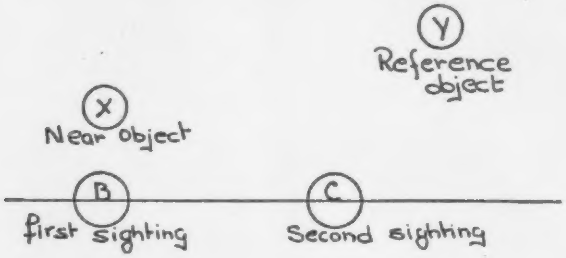
Y
Reference object
X
Near object
B
C
first sighting
Second sighting

Figure 3.7

6. Place the meter stick along the reference direction marked on your paper. Again be sure the edge is against pin A. Now turn the whole paper until you can again sight on the reference object (Y). This will orient your paper in the same direction for the two sightings.
7. Once again keep the paper in position, and, with the meter stick still against pin A, rotate the eyepiece end until you can sight on X. Draw the sight line along the edge of the meter stick. Call this line 1_2.
8. Before you move your paper or the meter stick, place another pin against the meter stick near the eyepiece (see D in Figure 3.8). Keeping the meter stick against pin D, rotate the stick until you can sight on the marker you left back at point B. Draw this third line and call it 1_3. Finally, measure the actual distance from where you are sighting (C) to the previous sighting location (B).

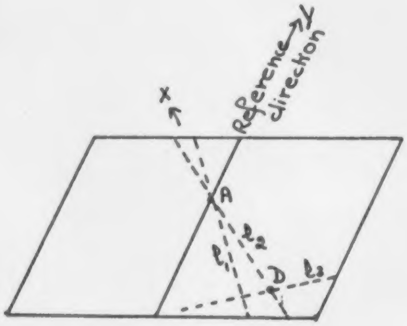

Figure 3.8

9. You now have two similar triangles. The triangle on your paper is formed by 1_1, 1_2, and 1_3 (Figure 3.8). The triangle we wish to measure is the "Parallax triangle" XBC (Figure 3.7). Use the known distance, BC, to calculate the distance BX and CX. Check your

results by measuring the real distances to the object at X.

Explain in your own words how the procedure you used here resulted in an indirect measurement of the desired distance.

Investigation 3.6

The method of parallax measurement can be used to determine many inaccessible distances, including the distance to another planet. However, for this case the indirect measurement is even more complex. In this activity we consider the problem of measuring the distance and direction to the planet Mars.

Our approach is that of the 17th century mathematician-astronomer Johannes Kepler, who is credited with determining that the orbits of the planets were elliptical, not circular. Kepler knew that if he could determine the distance and direction from Earth to Mars at a sufficient number of points, he could plot the orbit of Mars. Determining the direction to Mars was a direct measurement which posed no serious problem. However, the only way to measure the distance was by using parallax. The problem with this, though, was that no two points on earth were far enough apart to observe a parallax effect.

But there was another possibility for this measurement. If it were true that the earth moves around the sun, then observations of Mars could be taken at two places in the Earth's orbit which were far enough apart to notice a parallax shift (see E_1 and E_2 in Figure 3.9). This method would work just fine except for one thing: Mars, too, is moving, so by the time Earth moves from E_2 to E_1, Mars also will have moved to a new position. Parallax measurements depend on the object remaining in the same position while the observer moves.

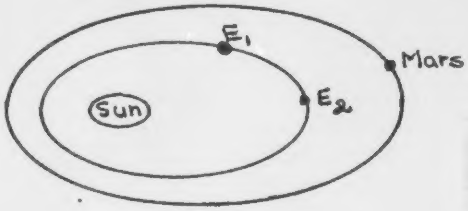

Figure 3.9

Kepler had two other sources of useful information, however: the orbit of earth could be plotted so it was possible to determine the position of Earth on any given date (see Investigation 4.4). Also, Copernicus had already determined that the Martian year was 687 days. This meant that every 687 days Mars completed one revolution around the sun. In this same time, Earth completed slightly less than two revolutions (since two Earth years equal 730 days). Thus, if Mars was observed when Earth was at E_1, then 687 days later Mars would be in the same place but Earth would be at E_2, 43 days short of returning to its initial position. This distance was enough to reveal a parallax shift. By noting the position of Mars against the

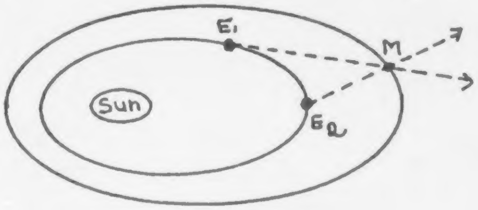

Figure 3.10

background of the fixed stars, one could determine the direction to Mars, and by calculating the position of Earth in its orbit, one could draw a diagram of these directions. The intersection of the two sight lines located the position of Mars in its orbit (see Figure 3.10).

You can apply these methods to plot your own orbit of Mars.

(1) Before you can plot the orbit of Mars, you first must plot the orbit of Earth. This can be done from observations of the sun as described later in Investigation 4.4. For the present investigation, however, you can approximate Earth's orbit by a circle with the sun at the center. Kepler's first law states that the orbits of the planets are ellipses with the sun at one focus. Since for Earth the ecentricity of the elliplical orbit is 0.017, the orbit is very nearly circular and the approximation is a reason able one. A circle of radius 10 centimeters is recommended.

(2) Table 3.1 gives eight pairs of observations taken 687 days apart. For each date, the positions of Mars and of the sun are recorded. These are given as geocentric longitudes--i.e., as directions from Earth to Mars or to the sun. In plotting the orbit of Earth, however, you must know the direction from the sun to Earth (heliocentric longitude) since the sun, not the Earth, is at the center of the orbit. But the direction from the sun to Earth is

just the opposite of the Earth-sun direction, or 180° more. Before you plot the orbit, convert the geocentric longitude of the sun into the heliocentric longitude of the Earth. The first pair is done for you.

(3) Use the heliocentric longitudes to locate the position of Earth in its orbit on each of the 16 dates. The convention being used here assigns 0° (the positive x-axis) to the direction from Earth to the sun on the vernal equinox (March 21). All measurements are made counterclockwise from that reference point.

(4) From the Earth's position at 1a and 1b, draw the direction to Mars on each date. To determine that direction, construct a line from the Earth position parallel to the axis XY as shown in Figure 3.11. This locates the 0° direction. The direction to Mars is measured counterclockwise from that reference. The intersection of the two lines locates one point on the orbit of Mars.

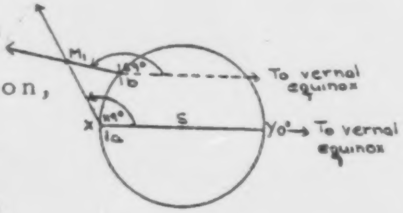

Figure 3.11

(5) Repeat the above for the other pairs of data to locate eight points on the orbit. Connect these points with a smooth curve. See if you can answer the following from your orbit:

Table 3.1[a]

Position	Date	Geocentric of Mars	Longitude of Sun	Heliocentric Longitude
1a	March 21, 1931	119.0°	0°	180°
1b	February 5, 1933	169.0°	315.7°	135.7°
2a	April 20, 1933	151.5°	29.4°	
2b	March 8, 1935	204.5°	347.0°	
3a	May 26, 1935	186.5°	64.7°	
3b	April 12, 1937	245.5°	21.7°	
4a	September 16, 1939	297.5°	173.5°	
4b	August 4, 1941	16.5°	131.1°	
5a	November 22, 1941	12.0°	238.8°	
5b	October 11, 1943	80.0°	198.3°	
6a	January 21, 1944	66.0°	300.5°	
6b	December 9, 1945	123.0°	255.4°	
7a	March 19, 1946	107.5°	358.0°	
7b	February 3, 1948	153.5°	313.7°	
8a	April 4, 1948	138.0°	13.7°	
8b	February 21, 1950	190.5°	331.9°	

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[a] Adapted from Harvard Project Physics, Preliminary Final Version, 1967.

Does the orbit which you plotted seem to support Kepler's conclusion that the planetary orbits are ellipses? If the distance from Earth to the sun is taken to be one astronomical unit (1 au), what is the distance from Mars to the sun at each of the eight points? What is the average Mars-to-sun distance? What is the closest distance (perihelion) and the farthest distance (aphelion) for Mars? As measured from the sun, what are the heliocentric longitudes of perihelion and aphelion for Mars? What is the approximate eccentricity of Mars' orbit? At what time of year is Earth closest to the orbit of Mars? What would be the minimum distance from Earth to Mars? From your orbit, can you verify Kepler's second law that the line joining the planet to the sun sweeps out equal areas in equal periods of time?

Teaching Notes

As noted earlier, each of the investigations in this chapter relies on indirect measurement. These are summarized below:

Investigation	Quantity to Measure indirectly	Accomplished through Measurement of
3.1	One very small unit	Many units
3.2	Time	Distance
3.3	Diameter of Earth	Shadow on Earth
3.4	Distance	Time
3.5	Distance	Parallax Angle
3.6	Position	Direction

The first investigation (3.1) differs from the others in that it is stated as a problem for which the student must devise a method. Other small quantities can be added or substituted for the ones suggested. Students should be asked to identify the underlying assumptions (such as the uniformity of the object to be measured) and the manner in which those assumptions affect the method. This activity also evokes questions about the precision of measurement and the probable error.

The other five investigations include quite specific procedures. Pupils should be challenged to explain in their own words how they are to make the various measurements and why they are doing each step. They should have a clear idea of what they wish to measure, why direct measurement is impossible, and how the indirect techniques yield the desired results.

Many suggested discussion questions are included in the investigations. Teachers can supply others, and both teachers and pupils are encouraged to offer other applications for the various techniques. Many of the activities described in Chapter 4 also involve indirect measurement. In addition to those, the following are some suggestions for developing other lessons:

1. Investigate different measuring instruments such as a voltmeter, barometer, and hydrometer, and describe how these devices work. Identify the quantity which is measured directly and show how it is used to determine the desired quantity.
2. How are echos used in measurement? How is the echo principle used in radar and sonar?
3. How are the following small quantities measured?
    - The size and mass of an atom
    - The size and mass of an electron
    - The thickness of a film of oil on water
4. How are the following large quantities measured?
    - The speed of a bullet, golf ball, tennis ball, etc.
    - The speed of sound
    - The speed of light
5. How are the following inaccessible quantities measured?

The temperature, mass, or size of a star

The height of a mountain or depth of a crater on the moon

The age of rocks or fossils

The age of a tree

6. Construct a sundial and explain how it is used to measure time.
7. How are the stars used for navigation?
8. How can one predict and measure an eclipse of the moon?
9. How can you measure the distance to the horizon?

Chapter 4

Ratio and Proportion

The patterns, relationships, and variations which we study frequently involve proportions. The measurements we make and the problems we solve often depend on our being able to determine the ratio of one quantity to another. In fact, ratios and proportions are among the most useful concepts in mathematics.

The concept of proportionality, which is associated with Piaget's stage of formal thought, develops later in a pupil's mental growth. Many of the activities in Chapter 2 illustrate simple proportions, and pupils should be guided to observe those cases where, for example, doubling one quantity causes another to double as well, or where doubling one quantity causes another to increase by a factor of four.

The investigations in this chapter aim to reinforce further the importance of ratios and proportions by showing their usefulness in solving problems. In many cases seemingly inaccessible solutions can be obtained from known data with the help of ratios.

Investigations

Investigation 4.1

Ratios play an important role in measuring the heights of very tall objects. Below are three simple techniques.

Study each and explain how it works. Describe the ratios involved in the measurement. Then construct the device and use it to measure the heights of objects like flag poles, steeples, tall buildings, TV antenna, etc.

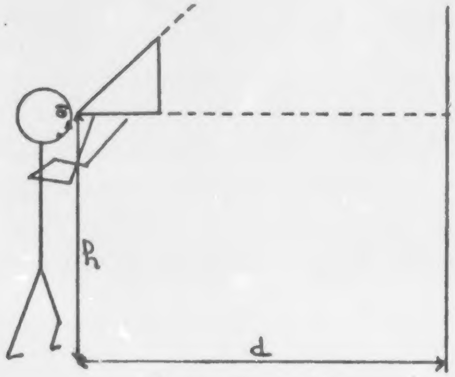

Figure 4.1

(A) Paper Triangles: Cut a square piece of paper and fold it along one diagonal. Describe the resulting figure.

Hold the triangle to your eye and sight along the folded edge. Have a partner observe you from the side to make sure you keep the bottom edge of the paper parallel to the ground. Holding the paper in that position, move backward or forward as necessary until you can sight on the top of the object. Measure the distance, d, from you to the object and the distance, h, from your eye to the ground. Explain why the height of the object is (d + h).

(B) Hypsometer: In the previous activity, the position of the paper triangle was fixed and you had to adjust your location accordingly. With the hypsometer you can stand at any convenient location and adjust the device to the necessary angle.

A simple hypsometer is made from a sheet of graph paper. A drinking straw is attached along one edge. At one end of the straw a string is attached with a weight at the other end. The string must be free to swing as it serves as a plumb line.

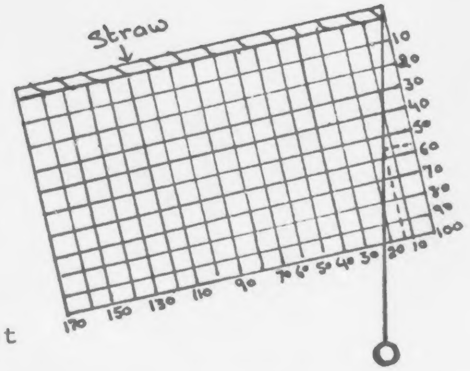

Figure 4.2

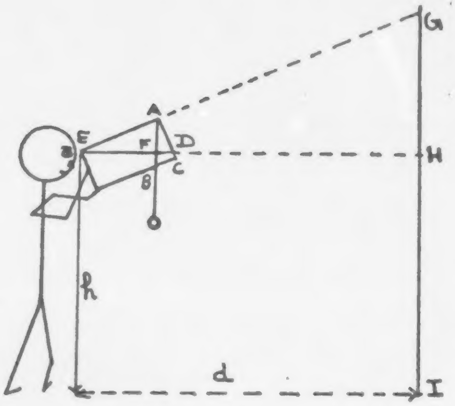

Figure 4.3

Sight through the straw at the object (see G in Figure 4.3). Note that $\triangle$ EFA on the hypsometer is similar to $\triangle$ EHG. Furthermore, $\triangle$ EFA is similar to $\triangle$ ACB ($\angle EFA = \angle ACB = 90^{\circ}$ and $\angle AEF = \angle FAD$ since both are complements of $\angle EAF$). Therefore, $\overline{BC}: \overline{AF}: \overline{GH} = \overline{AC}: \overline{EF}: \overline{EH}$. This allows you to use the scale on the hypsometer to determine $\overline{GH}$ if you know $\overline{EH}$. But $\overline{EH}$ is just the distance, d, from you to the object. For the reading shown in Figure 4.2, if d is 55 meters, $\overline{GH}$ is 15 meters. (What would $\overline{GH}$ be if d is 80 meters?) Then the total height is ($\overline{GH}$ + h), where h is the distance from your eye to the ground.

For further investigation, experiment with your hysometer to see how you can use it to measure the depth of a depression (distance from an upper floor window to ground, for example) as well as the height of an elevation.

More sophisticated hypsometers also have protractor scales printed on them which allow you to read directly the sine, cosine, and size of the angle of elevation. A simplified diagram is shown below. Study Figure 4.4 and explain how to read the sine, cosine, and tangent of the angle. Can you read the tangent from the hypsometer in Figure 4.2?

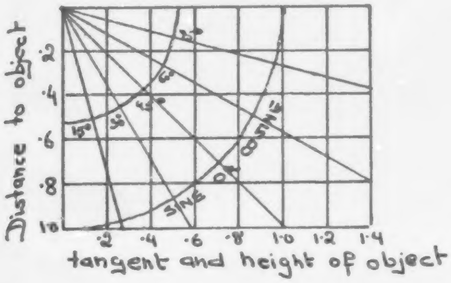

Figure 4.4

(C) Clinometer: This third device, which consists of a protractor and a viewer, also measures the angle of elevation to the object. Hold the protractor at eye level and rotate the viewer to locate the object of interest. The calculations for the height of the object are similar to those in the

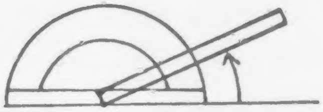

Figure 4.5

previous activities. Draw a diagram to represent this situation and describe the quantities which must be measured and the calculations to be made.

Investigation 4.2

You can construct a simple device for measuring small objects by copying the model shown below.

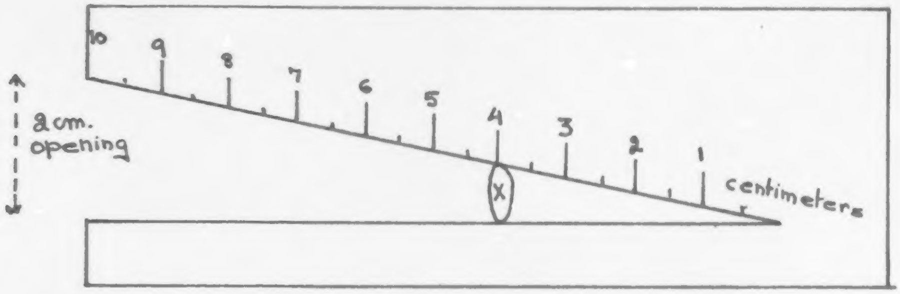

Figure 4.6

The object to be measured is inserted in the gauge. The drawing shows an object (X) whose height is 0.8 cm. Why? Explain how the gauge works and use yours to measure 10 small objects. Could you modify the scale or the gauge in some way so that you could read the height of X directly without any calculations?

Investigation 4.3

Perhaps you have heard the "logic" that the moon is only as large as a quarter because a quarter held close to the eye covers the moon. Or you may have smiled at the way cars and boats appear to be toys when viewed from an airplane. Experience gives us many examples of the way objects appear different at various distances.

Ordinarily we are able to compare objects with other things in the surroundings. If, however, we view two boats far out on the ocean with nothing around them to aid our judgment, the smaller one may actually appear larger only because it is nearer. Or the boats may seem to change their sizes because they are moving either toward or away from us. In this activity you will investigate the manner in which size appears to vary with distance.

Before you can begin, make a sighting device similar to the alidade in Figure 3.4a. This time, however, you will need to attach a scale to the front sight. This could be a small ruler, a strip of graph paper, or marks drawn by you. The actual size of the scale is not crucial, but the uniformity of the markings is. Also, you must be able to see the markings when viewing the object through the peephole. The smaller the scale divisions, the more accurate your measurements will be. It is recommended that the zero on your scale be about 1 cm above the ruler.

Select four or five objects of different sizes. For each, make a series of 10 or 12 observations at different distances. In a table, record the actual distance from the peephole to the object and the apparent size of the object as measured on the scale. Graph the apparent size as a function of the distance from the observer.

If an object, A, is actually twice the size of another object, B, how far must A be to appear the same size as B when B is at one meter? Compare different pairs of objects which you measured. If two unequal objects appear to be the same size, how do the ratios of their distances compare with the ratios of their sizes?

Have your partner hold an object of unknown size at a fixed distance. Measure the apparent size and try to determine the actual size. How close can you come? Measure the apparent size of the moon, then find the distance at which an object of known size appears to be the same size as the moon. Use your measurements and the known diameter of the moon (2160 miles) to calculate the distance to the moon.

Investigation 4.4

You can use the relationship between apparent size and distance which you determined in Investigation 4.3 to plot the orbit of the Earth. The procedure is described below.

The method of plotting directions in space is described in Investigation 3.6. Table 4.1 gives the geocentric longitude of the sun on various dates. Convert these to heliocentric longitudes and plot the direction from sun to Earth on each date.

Pictures of the sun were taken from the same observatory on each of the given dates. Measurements of the size of the sun in each picture showed variations throughout the year.

One possible explanation for this could be a periodic change in the volume of the sun, but astronomers have sufficient reason to rule out this alternative. That leaves as the best hypothesis the assumption that the distance from Earth to the sun is changing throughout the year.

Use your knowledge from Investigation 4.3 and the data in Table 4.1 to locate the distance between the sun and the Earth on each given day. Plot Earth's orbit. Does your orbit appear to support Kepler's three laws? (See Investigation 3.6.) In Investigation 3.6 you approximated Earth's orbit by a circle. Does that approximation appear to be justified? Would your orbit of Mars vary significantly if you plotted it about this Earth orbit?

Table 4.1[a]

Date	Geocentric Longitude of Sun	Apparent Size of Sun (cm)
March 21	0°	
April 6	15.7°	48.7
May 6	45.0°	48.7
June 5	73.9°	48.5
July 5	102.5°	48.1
August 5	132.1°	48.6
September 4	162.0°	49.0
October 4	191.3°	49.5
November 3	220.1°	49.7
December 4	250.4°	49.9
January 4	283.2°	50.0
February 4	314.7°	49.6
March 7	346.0°	49.5

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[a] From Harvard Project Physics, preliminary final version, 1967.

Investigation 4.5

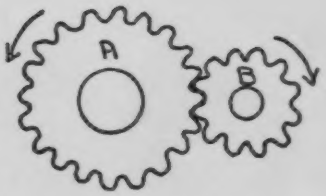

Figure 4.7

The toothed gears shown in Figure 4.7 have different numbers of teeth so that A makes three revolutions in the time B makes 5. Investigate the gear ratios for different gears. To do this you can use gears from various children's toys, or you can construct cardboard models from circles of different diameters. Determine a relationship between the number of teeth and the number of revolutions of the gears. Try to extend your relationship to three or more gears.

Investigation 4.6

In the previous investigation you found a ratio which described the numbers of revolutions for gears with different numbers of teeth. One familiar object which depends on gears is a bicycle. In this activity you can investigate the role of gears and gear ratios in the operation of a bike.

(A) Before you investigate the more complex gear-driven bikes, study the simpler case of the tricycle. In a tricycle, the pedals are attached directly to the front wheel. When the child's feet make one revolution, how many revolutions does the front wheel make? Measure the radii of the pedal and the wheels of a tricycle and determine how far the feet travel in one revolution. How far does the front wheel travel during that same time? How many turns must the rear wheels make during this same time? Will the answers be the same for all tricycles? If not, what factors would account for differences? Try to observe a child riding a tricycle and determine the average speed of the child's feet and the average speed of the bike.

(B) Next study a single-speed bicycle in which the pedals, near the center of the bike, are attached by a chain to the rear wheel. Again find the ratio of pedal revolutions to wheel revolutions. Determine the radii of the pedals, the wheels, and the two gears (one attached to the pedals and one at the hub of the rear wheel). Also count the teeth in each gear. Is there a relationship between gear sizes, numbers of teeth, and distances traveled by feet and wheels? Compare these findings for bikes of various sizes. Try also to determine the average speed for this kind of bike.

(C) A ten-speed bike has two front pedal gears and five rear wheel gears, each with a different radius and number of teeth. Different combinations of front and rear gears produce different pedal-to-wheel speed ratios. When the shift lever is moved, a derailleur pushes against the side of the chain and derails it from one sprocket to another. The lower gears, those with low-speed ratios, are desirable for uphill climbs; the greater speed ratios of the higher gears are used for racing. An example of the gear sizes for a ten-speed bike is as follows:

Number of teeth on pedal gears: 39, 51

Number of teeth on wheel gears: 14, 17, 20, 24, 28

Diameter of wheel: 27 inches (68.6 cm)

Count the teeth on each gear of your ten-speed bike and compute the gear ratios for each of the ten combinations of front and rear gears. Make a chart to show the combinations to use and the resulting ratios for each of the ten gears. Remember that first gear has the lowest ratio. Also determine the distance traveled by the bike in one revolution of the pedals for each gear and include that information in your table. If you always pedal at the same rate on level ground, what average speed could you expect to attain for each gear?

(D) Now look at a three-speed bike. You will notice that the wheel gears are enclosed within the sealed hub where you cannot see them. Using the principles you learned in the earlier investigations, determine the pedal-to-wheel ratios and the number of teeth on the pedal gear, and predict the numbers of teeth on the hidden gears.

Investigation 4.7

Use the solar system statistics given below to make a scale model of the solar system.

Solar System Statistics

Planet	Symbol	Average distance from the Sun			Equatorial diameter			Period of revolution, years	Period of rotation, days
		Million kilometers	Million miles	Astronomical units	Kilometers	Miles	Relation to Earth		
Mercury	☿	57.9	36.0	.387	4,880	3,032	.38	.24	58.60
Venus	♀	108.2	67.2	.723	12,100	7,519	.95	.62	243.00
Earth	⊕	149.6	93.0	1.00	12,756	7,927	1.00	1.00	1.00
Mars	♂	228.0	141.7	1.52	6,784	4,215	.53	1.88	1.03
Jupiter	♃	778.0	483.4	5.20	143,200	88,984	11.20	11.86	.41
Saturn	♄	1,425.0	885.5	9.52	120,000	74,568	9.41	29.46	.43
Uranus	⛢	2,867.0	1,781.6	19.20	51,800	32,189	4.10	84.01	.57
Neptune	♆	4,486.0	2,787.6	30.00	49,500	30,759	3.90	164.10	.77
Pluto	♇	5,890.0	3,660.0	39.4	2,500	1,616	.20	247.00	.39
Sun	☉	—	—	— 11	$1.4(10^6)$	$8.7(10^5)$	—	—	—
Nearest Star		—	—	$2.7(10^5)$	—	—	—	—	—

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[a] Planetary data from Air and Space, November-December 1979.

If the sun is represented by a basketball, what objects could be used to represent Earth? Jupiter? Pluto?

If Mercury is placed 1 cm from the sun, how far away would Pluto be placed? Where would you place the nearest star?

Discuss the difficulties in representing both size and distance in the same scale model.

Investigation 4.8

A popular advertising campaign features the Jolly Green Giant and his friend Sprout. Both the giant and Sprout appear very human-like, except one is much larger and the other much smaller. Suppose the giant is ten times the height of a man (say 20 meters for convenience), and suppose Sprout is one-tenth human size (say 20 cm). Consider the likelihood of these creatures actually existing.

(a) Since both the giant and Sprout appear to be proportioned like humans, assume that not only their height but also their width and thickness are in the given ratios. How will the cross-section of their bones compare with a human's? Since the body's strength is proportionate to the cross-section of the bone, how will the strength in the legs of each compare? Weight, however, is proportionate to volume. What will be the ratio of weight to strength for each of these creatures compared to a man's? What is the likelihood that the Jolly Green Giant can stride across the valley? In a tug-of-war, how many Sprouts would be needed to balance the pull of the giant?

(b) How does the surface area of Sprout's body compare with his friend the giant's and with yours? How does the weight compare? When an object falls, the resistance to falling caused by the air is proportional to the surface area of the falling body. How does this enable Sprout to jump freely from the Giant's tall cornstalks.

(c) When you climb out of a swimming pool, the water which clings to your body is about .01 cm in thickness. If Sprout accidentally lands in a puddle, how does the weight of water on his body compare with his body weight? Will he be able to survive without the giant's help?

(d) Assuming that the amount of food needed to sustain body functions is proportional to the body's mass, how much of his vegetable crop will the giant consume compared to the humans he feeds? How much will Sprout eat? There is another problem, however. Creatures lose body heat through their skin, and to maintain body temperature warm-blooded creatures need food proportionate to their body surface. How would Sprout's food consumption rate compare to yours if he is to survive the chilly nights in the valley?

Teaching Notes

Of major interest in these investigations are the relationships between quantities which are in a fixed ratio. From these we are able to determine proportions which enable us to arrive at desired new information. In doing the investigations, students should identify the relevant proportions and explain in their own words how variations in certain quantities affect others.

Investigation 4.1, which involves indirect measurement of heights, suggests three techniques. Many others are possible, and students can be challenged to research some or to develop their own. The ones described here require only elementary concepts; pupils with some knowledge of trigonometry can find many more techniques involving the trigonometric ratios.

Investigations 4.3 and 4.4 show an example of a problem which must rely on ratios and proportions since only indirect measurements are possible for astronomical distances. The data also illustrate how even seemingly small variations can be significant when scaled to actual magnitudes.

Investigations 4.5 and 4.6 relate the usefulness of ratios to a subject of considerable interest to students: bicycles. The questions included in the activity probably will stimulate more questions from the pupils. Related investigations can be developed from these. For instance, how do the gear ratios affect the force exerted by the bike against the ground? Why are low gears needed for uphill rides? How does the size of the rider affect the operation of the bike?

Investigations 4.7 and 4.8 generally intrigue pupils and are especially useful in motivating class discussion. The model solar system is not difficult to construct for either size or distance, but only when pupils attempt to represent both on the same scale do they realize the significance of the problem. The questions about scaling also generate enjoyable discussions. Pupils should be encouraged to read the delightful essay "On Being the Right Size" by J.B.S. Haldane (in Newman, The World of Mathematics, Vol. II). Some of the interesting investigations which can result are suggested below.

1. Investigation 4.8 involves scaling, an important consideration in many physical and biological situations. Investigate other situations in which scaling is significant. Some suggestions are as follows:
    a. As animals such as the horse evolved, they got larger. What are some of the implications for such things as skeleton, organs, food, oxygen requirements, heartbeat, blood pressure, etc.?

b. Look up the sizes believed to be attained by some of the ancient reptiles. For creatures this size, what are some of the requirements for sustaining life?

c. How do the rates of food, oxygen, and water consumption of very small animals like mice compare with man's or a larger mammal's?

d. Why do small animals have eyes which appear to be so much larger in relation to their heads than do very large animals?

e. How do scaling considerations affect the construction of buildings? For example, could a building like the Empire State Building or the Sears Tower be enlarged twice? ten times?

f. In order to keep a plane in flight, a certain minimum speed must be maintained. That speed varies with the square root of the plane's length for an airplane of given design. If a plane's dimensions are increased by a factor of four, how would this affect the required minimum speed? How would it affect the volume and weight of the plane? Therefore, how would it affect the power needed to produce that speed?

g. Moviemakers filming horror or disaster movies or outer space spectaculars make use of scale models. For example, to film a giant lizard falling from a skyscraper, a real lizard may be dropped from a model of the building. Suppose the scale model is only

1/100 the height of the actual building. How will this affect the time of fall? If the movie will be projected at the rate of 24 frames per second, at what speed should the sequence be filmed in order to create the proper illusion of time? How might you film a building collapsing during an earthquake? The Starship Enterprise at warp speed?

2. Make a map of a designated area of land.
3. Make a topographic profile of a designated region.
4. Two techniques for enlarging or reducing drawings are with grid paper and with an instrument called a pantograph. Investigate how each of these works and use the method to enlarge or reduce a picture.
5. In Investigation 2.12 you made a flip book to represent a motion. What happens if you make two copies of each page in your book and arrange the pages so each pair of identical pages is together? What if you make three copies of each page? four?
6. The instruments we use to measure often use ratios. For example, the expansion of a liquid when heated is magnified, as in a thermometer, by forcing it to rise in a very narrow capillary. Determine how the height of the liquid in the tube is affected by the diameter of the tube.

7. Another way to magnify very small motions, such as the expansion of a metal rod, is to make the expanding rod cause a pin to roll. Attached to the head of the pin is a large circular dial, perhaps 10 cm in diameter. How would you use ratios to determine how far the rod expanded if you noted that the dial turned through 20°?

8. Investigation 4.1 showed three ways to measure heights by indirect means. Find some other ways to make indirect measurements. Some suggestions include:
   a. How can shadows be used to measure height?
   b. If you look into a puddle directly in front of you and see reflected there a treetop from the far side of the puddle, how can you determine the height of that tree?
   c. How can you measure the distance across a lake, river, or freeway which you cannot cross?

9. The frequencies of the notes of a musical scale are in fixed ratios to each other. Find these ratios for one or more scales. Also, investigate at least one string instrument, at least one wind instrument, and at least one percussion instrument, and explain how each is able to produce different tones. What is varied in order to change the tone? Is there a relationship between that quantity and the ratios of the frequencies of the notes?

Chapter 5

Spatial Relationships

The previous activities investigated patterns and relationships among variables in a variety of situations. In general, these were expressed in numerical relationships of one kind or another. However, many of the relationships of mathematics are spatial in nature and involve location, position, size, shape, orientation, symmetry, similarity, congruence, etc. These relationships, too, are encountered in many applications of science.

For the activities described in this chapter, pupils must locate points or objects; compare the size, shape, orientation, or position of geometric figures; describe properties of geometric figures; determine relationships between figures or among parts of a figure; find symmetries and, in general, apply concrete and intuitive ideas to concepts of location, orientation, size, shape, and distance. The activities also apply numerical methods to describe certain geometrical relationships and patterns. These activities allow the students to develop and reinforce geometrical concepts through action on concrete objects, and they further demonstrate that the patterns of mathematics are not just abstract numerical creations but that they exist as well in real-world phenomena.

Investigation 5.1

We know from experience that when we look in a plain mirror we see an image of ourselves which appears to be standing somewhere behind the mirror. We can study this phenomenon with the help of a MIRA. The MIRA is a piece of plexiglass with perpendicular end pieces which allows it to stand upright.

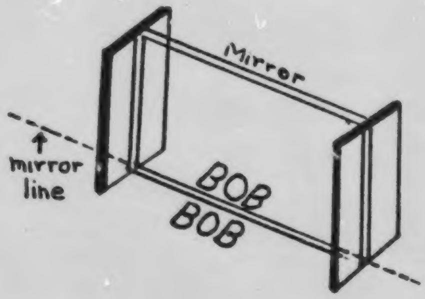

Figure 5.1

To use the MIRA, draw a line across the center of a piece of paper. This is the mirror line. The beveled edge of the MIRA is placed on this line. The commercial MIRA is in correct position when the words MIRA-MATH in the upper left corner are in position to be read. Draw a mirror line and position the MIRA. Then print your name on the paper in front of the MIRA. Look through the MIRA and you should see the image of your printing behind the MIRA. Unlike the ordinary mirror, however, the MIRA allows you to trace an image behind the surface while you look through the MIRA. Trace your name as it appears in the MIRA. Make several other figures and practice tracing their images. Then investigate the following:

(a) Draw about ten figures at various distances from the MIRA. The figures should vary in size, shape, and distance from the MIRA. Trace the images of each. Describe fully the relationship between the original (object) and the image. Be sure to account for the position of the image, its distance from the MIRA, the size of the image, and the orientation of the image.

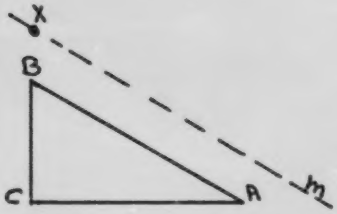

Figure 5.2

If ∆ ABC is a 30-60-90 degree triangle and m is the mirror line, describe the image A'B'C' by answering the following:

How do angles A, B, C in the object compare in size to angles A', B', C' in the image?

How do the lengths of the sides of ∆ ABC compare to the lengths of the sides of ∆ A'B'C'?

How do the distances of points A, B, and C from line m compare to the distances of points A', B', and C' from line m?

How does the position (orientation) of ∆ ABC compare with the position (orientation) of ∆ A'B'C'?

(b) For ∆ ABC described in Figure 5.2, begin with the MIRA on line m and slowly move the MIRA away from and then toward the triangle. Always keep the MIRA parallel to line m. Describe the size, shape, location, and orientation of the image as the distance from ∆ABC to the MIRA changes. Repeat this with several other figures.

State a generalization which describes the relationships between the size, shape, location, and orientation of the object and its image as the distance to m changes.

(c) Mark a point X on line m (see Figure 5.2). Repeat (b) above but this time keep the MIRA on the paper while you rotate it about point X. State a generalization which describes the relationships between the size, shape, location, and orientation of the object and its image as the MIRA rotates about a point.

(d) Place the MIRA on line m as in Figure 5.2 and slowly tilt the top edge of the MIRA toward, then away from the object. Describe the relationships between the size, shape, location, and orientation of the object and its image as the plane of the MIRA rotates about line m.

(e) A line of symmetry in a figure divides the figure into two congruent parts. In other words, the figure can be reflected onto itself if the line of symmetry is the mirror line.

Draw the following:

square
rhombus
rectangle (not square)
parallelogram (not rhombus)
regular hexagon
regular octagon
trapezoid
isosceles trapezoid
isosceles triangle
equilateral triangle
regular pentagon
scalene triangle
circle
kite (2 pair of equal adjacent sides in this quadrilateral)

Use the MIRA to show all the lines of symmetry in each of the above. Which have only one? two? three? more? something else? Collect some leaves from different plants and use the MIRA to determine the symmetries in the leaves.

(f) A figure has point symmetry if it can be rotated about the given point and made to correspond to itself. For example, the center of a square is a point of symmetry because a rotation of 90° about that point maps the square onto itself. What other rotations make the square correspond to itself? Cut out of paper two congruent models of each of the figures listed in (e) above and determine if each has point symmetry by placing one of the cutouts on the other and then rotating the top figure. If it has rotational symmetry, state both the point (center) of symmetry and the degrees of the rotation.

(g) Extend the idea of a line of symmetry (for a plane figure) to a plane of symmetry (for a solid). Place different solids such as a cube, rectangular prism, cone, cylinder, other prisms, tetrahedron, octahedron, etc., in front of the MIRA. Describe their images. Use a piece of thin cardboard to illustrate the planes of symmetry in the images of these solids. Use models of various crystals and determine their planes of symmetry.

(h) Transformational geometry is an approach to geometry which studies the behavior of points and lines under certain rigid motions or transformations. The three basic transformations are translations, reflections, and rotations (sometimes called slides, flips, and turns, respectively). Look up each of these and write a definition in your own words. Use your MIRA to demonstrate the definitions you wrote. Can all of these transformations be achieved with a MIRA? How are an object and its image under each transformation related?

(i) Cut two congruent 30-60-90 degree triangles from a piece of paper. Label one "object" and the other "image." Randomly toss the triangles onto the desk. Use the MIRA to show that no matter where the triangles land, it is always possible to reflect the object onto the image. (It may be necessary to use more than one reflection.) State some rules which describe where to place the MIRA if:

(1) the image is a reflection of the object
(2) the image is a rotation of the object
(3) the image is a translation of the object
(4) the image is a combination of a reflection and a translation

(5) the image is a combination of a rotation and a reflection

(j) Use your MIRA to make the following constructions. In each case describe briefly how you did the construction.

parallelogram
rectangle
rhombus
square
isosceles triangle
equilateral triangle
regular hexagon
regular octagon
midpoint of a line segment
perpendicular bisector of a segment
perpendicular from a given point to a given line
line parallel to a given line through a given point
bisector of an angle
circle congruent to a given circle and tangent to a given line
altitudes of a triangle
medians of a triangle
center of a circle

(k) Explain how you could use the MIRA for the following:

(1) to determine if a chord is a diameter

(2) to test if two lines are parallel

(3) to test if two lines are perpendicular

(4) to test if a triangle is isosceles or equilateral

(5) to test if a quadrilateral is a trapezoid, parallelogram, rectangle, rhombus, or square

(l) Use your MIRA to "demonstrate" the following theorems:

(1) the base angles of an isosceles triangle are equal

(2) the diagonals of a rectangle bisect each other

(3) the diagonals of a rhombus are perpendicular

(4) every translation is a composition of reflections

(5) every rotation is a composition of reflections

(6) the segments joining the midpoints of consecutive sides of any quadrilateral form a parallelogram

Investigation 5.2

Consider the situation in the diagram in Figure 5.3. The triangle in front of the MIRA is reflected and its image is traced behind the MIRA. An observer at point P looks at the image of the triangle. The sight line from P to the right angle of the triangle is shown. It appears to the observer as though a triangle being viewed is located behind the MIRA. In reality, however, light traveling from an object to a mirror would be reflected at the surface of the mirror and would travel to the observer along the path shown (solid line). The line l is the perpendicular to the mirror line at the point of reflection. The angles between the incident light and l and between the reflected light and l are called the angles of incidence (i) and reflection (r) respectively.

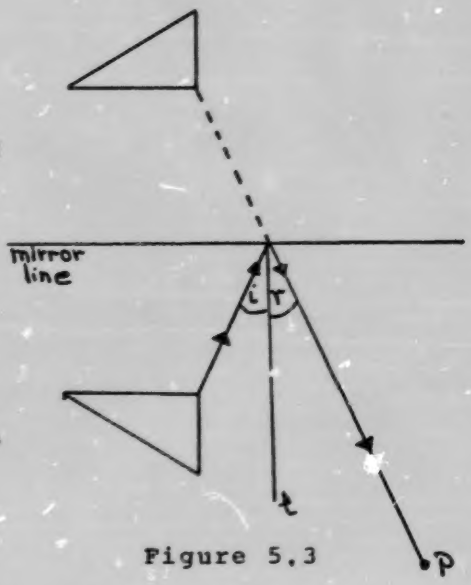

Figure 5.3

Make a triangle similar to the one in the previous diagram. Pick a point P and place a pin at that point. Line your ruler against the pin and sight along the ruler the image in the MIRA. (You will want to select a definite point on the image to sight on--or, better yet, repeat the sighting several times using different points such as the three vertices.) Draw the sight line. Connect the corresponding point on the object to the point of intersection of the sight line and the mirror line. Measure the angles of incidence and reflection.

Repeat the above with several more figures, varying the distance and position of the objects as well as the position from which you sight. What can you conclude about the angles? What happens if the sight line is perpendicular to the mirror line? State a generalization about the angles of incidence and reflection when an object is reflected from a plane mirror.

Investigation 5.3

Draw a line on the paper. Using two mirrors at an angle to each other, explore what happens when the angle between the mirrors changes. Can you make a triangle? square? pentagon? hexagon? octagon? other? What relationships appear to hold between the figure made and the angle between the mirrors? What happens if the two mirrors are parallel to each other?

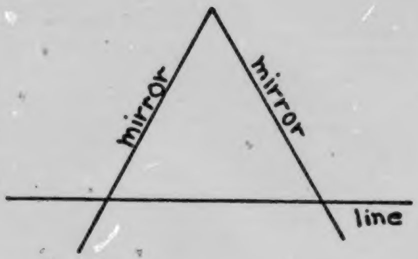

Figure 5.4

Experiment with three (or more) mirrors formed into an enclosed area (for example, three mirrors making a triangle). Place various colors and shapes into the kaliedoscope thus formed. Explain what is happening.

When a shape which is repeated indefinitely completely covers a surface, the shape is said to tessellate the surface. (An example is square tiles covering a floor.) Is it possible to tessellate the plane using triangles? using quadrilaterals? using pentagons? hexagons? octagons? Will only special polygons tessellate? If so, which ones? Prove the claim that any triangle and any quadrilateral will tessellate the plane.

Investigation 5.4

When light travels through different materials it travels at different speeds. In some materials it travels faster than in others. Hence, when the light passes from one medium to another, its speed changes and the path of the light may appear to bend. A common experience of this is the illusion of depth you get when you look down into a clear pool.

For this experiment you will use a small plastic box filled with water. Place the box on the paper and trace its outline. Now remove the box and draw a line perpendicular to the far edge of the box. Place two pins in the paper along this line.

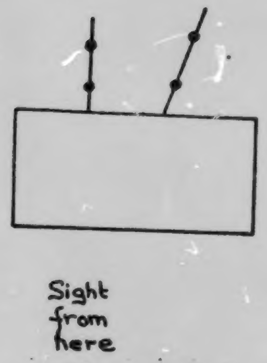

Figure 5.5

Now sight through the box at the pins. In the paper between you and the box place two more pins so that the four pins all line up. Remove the box and draw a line adjoining the two pins you just placed. How does this line compare with the line along which the original pins were placed?

Repeat the experiment several more times, but these cases should have the original pins along lines oblique to the box. Again, place two pins between you and the box so that all four appear to line up. Draw the line through the last two pins to the near edge of the box. Then draw the segment through the box, joining the points where the two lines meet the box edge. This is the path of the light. Repeat all of the above, using boxes of other shapes such as triangular prisms.

Define the angles of incidence and refraction as before--i.e., the angles between the light and the perpendicular. What generalizations can you make about the path of refracted light?

Investigation 5.5

When light strikes an object, it is reflected in all directions. The light which reaches your eye enters through the iris, which automatically opens to the correct width to admit exactly the right amount of light. The lens in the eye focuses the light so that an image of the object is formed on the retina at the back of the eye. The brain interprets this sensation so that you "see" the object.

A camera operates in essentially the same way. The amount of light entering the camera is controlled by adjusting the shutter speed and the size of the aperture. A lens focuses the light to form an image on the back of the camera. A film treated with light-sensitive chemicals replaces the retina, and the image is produced when light reaching the film causes chemical changes which result in a permanent imprint of the object. Developing the film translates the chemical changes into a photographic image or print.

This investigation and the next explore the relationship in this process. Here you will consider the simplest kind of camera, a pinhole camera in which light is focused through a small hole rather than through a lens. For the investigation you need a candle, a piece of white poster board, and a piece of cardboard about the size of a standard sheet of typing paper. You also may need to darken the room.

(1) Using a compass point, make a small, very round hole about 3 mm in diameter in the piece of cardboard. Stand the candle, the cardboard, and the poster board in a line as shown in Figure 5.6. Keep the distances small.

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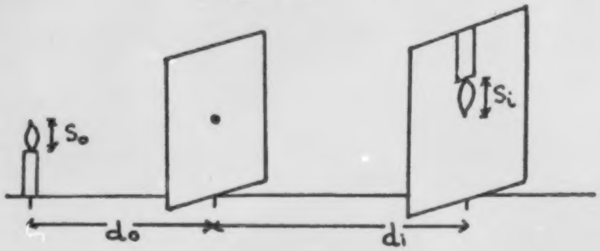


Figure 5.6

Adjust the pieces so an image of the candle flame is formed on the poster board. Change the distance (d_i) from the pinhole to the image. What effect does this distance have on the image? Change the distance (d_o) from the candle to the pinhole. What effect does this have?

(2) For at least six different relative positions of the three pieces, measure d_o: the distance from candle to pinhole; d_i: the distance from pinhole to image; s_o: the height of the candle flame; and s_i: the height of image flame. Record your data in a table as shown below:

object distance d_o	image distance d_i	object size s_o	image size s_i	magnification s_i / s_o	ratio d_i / d_o

(3) What generalizations can you make from the information in the table? When is the image of the flame larger than the real flame? When is it smaller? Verify your conclusions by testing a few more values of d_o and d_i. If a given value for d_i

produces an image 10 cm high (s_i = 10 cm), where should the screen be to form an image with s_i = 20 cm? with s_i = 5 cm?

Make a drawing to explain the results you obtained. (Hint: Use similar triangles.)

Investigation 5.6

Now replace the pinhole with a lens such as a magnifying glass or an eyeglass lens. The lenses in this investigation will be convex.

(1) First take the lens to the window and move the screen (poster board) until you find a clear image of a distant object such as a tree or tall chimney. Find the position for which the image is as sharp as possible. This distance from the lens to the screen is the focal length which we will call f. Record f for your lens. Also calculate $\frac{1}{f}$.

(2) Now light the candle and find the position of the lens that produces a sharp image of the candle flame on the poster board. Describe the image flame.

(3) Place the candle and the poster board within about 2 meters of each other. Find a position of the lens that produces a sharp image. Are there other lens positions that also produce sharp images? For at least ten different relative positions,

measure d_o, d_i, s_o, and s_i. Be sure to include some cases for which $d_o > 2f$, some for which $f < d_o < 2f$, and one for $d_o = 2f$.

Complete this table:

$1/f =$ _______________.

object distance	image distance	object size	image size	magnification	d_i/d_o	$1/d_o$	$1/d_i$	$1/d_o + 1/d_i$

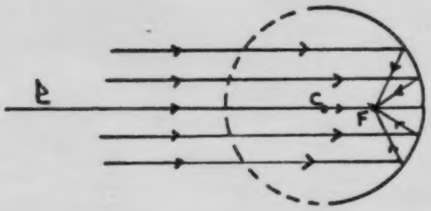
Does the table suggest a generalization relating d_o and d_i? What can you say about magnification in this activity? Graph d_i as a function of d_o and write an expression to describe this function.

(4) Remove the screen and the candle and place a long straight pin behind the lens at a distance $d_o < f$. View the image of the pin by looking through the lens at it. At the same time, have your partner hold a thin pointer (a pencil will do) above and behind the lens so that you can see the pointer and the pin at the same time. When the pointer (seen directly) and the pin (seen through the lens) show no relative shift when you move your head, the pointer then locates the image of the pin. Record this distance (d_i).

Repeat for several more values of $d_o < f$ and add your data to the table in (3) above. Do your earlier conclusions apply for cases of $d_o < f$? If not, what is different?

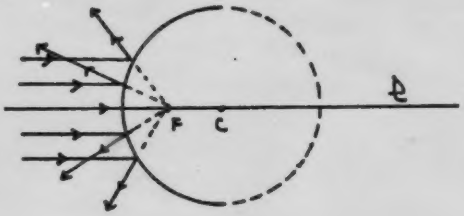
Investigation 5.7

Several earlier investigations concerned the behavior of light when it is reflected from a plane mirror. Another important type of mirror is the curved reflector used in such things as search lights. These are commonly constructed using a part of the surface of a sphere for the mirror. A reflector made from the inside of the sphere forms a concave mirror; the outside of the sphere produces a convex mirror.



Concave Mirror

Figure 5.7a



Convex Mirror

Figure 5.7b

Figure 5.7 illustrates these two cases. Both represent the curved mirror as the arc of a circle whose center is C. The line through C perpendicular to the center of the mirror (line 1) is called the principal axis of the mirror, and C is known as the center of curvature. Figure 5.7a shows what happens when the light rays parallel to the principal axis strike the concave mirror. Since the light obeys the law of reflection, the angle of incidence equals the angle of reflections, and the light is reflected back through the midpoint of the segment from C to the mirror. This point (F) is called the principal focus and the distance from F to the center of the mirror is known as the focal length of the mirror. In the case of the convex mirror (Figure 5.7b), the reflected light diverges as shown, but the light rays appear to come from a point behind the mirror. In this case, F is called the virtual focus of the mirror.

(1) Construct several arcs of varying radii and use these to verify the claims in the above paragraph. To do this, trace the paths of several light rays parallel to the principal axis. At each point of reflection, construct an angle of reflection equal to the angle of incidence to determine the path of the reflected rays. Show that for concave mirrors the reflected rays are concurrent at a point (F). Compare the focal length to the radius of curvature. For convex mirrors, extend the paths of the reflected rays behind the mirror to establish their concurrence at the virtual focus. Again compare the focal length to the radius of curvature.

(2) Consider the two curved mirrors shown in Figure 5.8. The vertical arrow in each drawing represents an

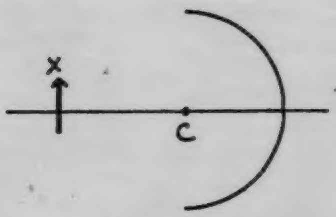


Figure 5.8a

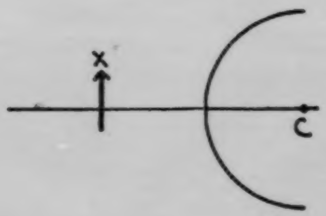


Figure 5.8b

object which is reflected in the mirror, and X is a point on that object. (For convenience, let X be the tip of the arrow.) For each of the following cases, construct the path of the reflected light in both mirrors:

(a) The incident light is a ray through X parallel to the principal axis.

(b) The incident light is a ray through X and C (i.e., the ray passes through the center of curvature).

(c) The incident light is a ray through X and F (i.e., the ray passes through the principal focus).

Show that the three reflected rays which you constructed in (a), (b), and (c) above are concurrent at a point X'. (Remember that for the convex mirror you must extend the reflected rays behind the mirror.) The point X' locates the image of X in the mirror.

(3) In (2) above, X was the tip of a vertical arrow perpendicular to the principal axis. Let the

other end of that arrow be called Y, and locate the image, Y', by the same process you used to find X'.

X'Y' now represents the image of XY in the mirror. Describe the size, location, and position of X'Y' relative to XY. Repeat the process of locating images for each of the following conditions:

(a) The object is at various distances greater than the radius of curvature. What happens to the image as the object moves in from infinity toward the mirror?

(b) The object is at a distance 2f, where f is the focal length. (Hence, 2f is the radius of curvature.)

(c) The object is at a distance between 2f and f.

(d) The object is at the distance f.

(e) The object is at a distance less than f.

(4) When the image appears to be in front of the mirror it is called a real image; an image which appears to be behind the mirror is called a virtual image. From your observations in (3) above, answer the following:

(a) Where will an object be located in order to produce a real image in a concave mirror?

(b) Where will an object be located to produce a virtual image in a concave mirror?

(c) Is the image in a concave mirror inverted or erect?

(d) Is the image in a convex mirror inverted or erect? Is this image real or virtual?

(e) Is the image in a convex mirror larger or smaller than the object?

(f) Is the image in a concave mirror larger or smaller than the object? Is the image ever equal in size to the object? If so, at what object position?

(5) For the various locations of the object which you have investigated, measure the following quantities and record the values in a table:

d_o: distance from object to mirror
d_i: distance from image to mirror
s_o: size of object
s_i: size of image
f: focal length of mirror

Also record the following reciprocals $\frac{1}{d_o}$, $\frac{1}{d_i}$, $\frac{1}{f}$.

(a) From your data determine the relationship between $\frac{1}{d_o}$, $\frac{1}{d_i}$ and $\frac{1}{f}$.

(b) From your data determine the relationship between the ratio of image size to object size and the ratio of image distance to object distance.

(6) Locate some objects in your school or home which use curved mirrors. Describe the functions of the mirrors in those objects.

Investigation 5.8

Recall that when you view an object such as your outstretched thumb first with the right eye and then with the left, the object appears to shift its position (the parallax effect). This is due to the separation between your eyes which causes each eye to view the object from a different position.

Suppose now that we reverse the process. Instead of viewing an actual object and noting the images of it which appear against a background, suppose we provide drawings of these images in such a way that each eye sees only the drawing which is intended for it. The message that would reach the brain from your two eyes viewing these drawings would be very similar to the message received if you viewed the real object. Hence, what results is an optical illusion of a three-dimensional object.

A picture in which two drawings are superimposed in such a way as to produce this illusion is called a stereogram. Figure 5.9 illustrates how the stereogram works. The point X represents the object you are viewing -- say the tip of your pencil. Points X_R and X_L are the images of X against the paper as they are viewed by your right (R) and left (L) eyes, respectively. Now suppose we color point X_R blue and point X_L red, and have you view the points through "glasses" having a red lens for your right eye and a blue lens for your left eye. Any light coming to the right eye must pass through the red filter. Now ordinarily "white" light is actually composed of light of many different wavelengths corresponding to the different colors, the familiar red, orange, yellow, green, blue, and violet of the spectrum. (You see this, for instance, in a rainbow because white light passing through the water droplets in the atmosphere is spread out or dispersed revealing the various colors.) In the case of the red lens, the filter material has the property of blocking out the red light while allowing the other wavelengths to pass through. Hence, the blue dot can be seen through the red lens (it should appear black), but the red dot cannot be seen because the red light does not reach the eye. In a similar manner the red dot only is seen through the blue lens. Thus, when you look with both eyes through the

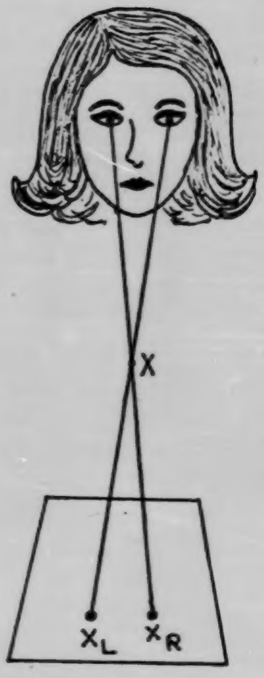


Figure 5.9

glasses, each eye sees only one dot, but your brain superimposes the images and interprets the message as if you were seeing the single dot at X in space. X is, of course, the intersection of the sight lines RX_R and LX_L.

(1) In this activity we will learn to make stereograms. First, however, we must make the viewers. This is described below.

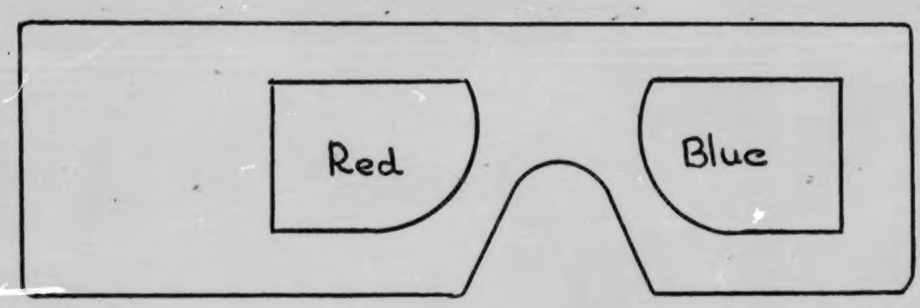
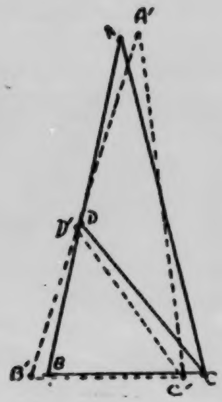
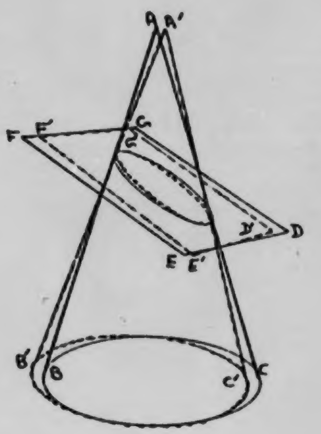


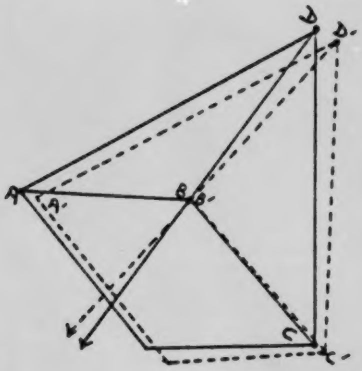
Figure 5.10

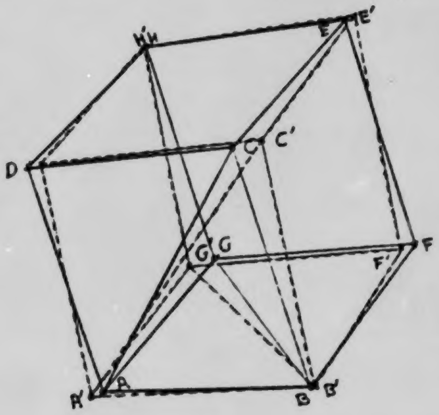
The pattern in Figure 5.10 should be a good size for a viewer for the average person. Cut two copies of the viewer from stiff paper or light cardboard. An index card is good for this purpose. The red and blue lens material is the film called "gelatin" commonly used to cover theater spotlights. It can be purchased from art supply stores at a few cents a yard. Cut a small piece of each color and place them over the eye holes. Glue the two viewer pieces together with the colored filters between.

After you have made the viewer, copy Figure 5.11 onto white paper. Make all solid lines red; make all dotted lines light blue. Place the paper on the table and view the drawings through the viewer. You will have to let your eyes adjust until the picture appears three-dimensional. This may take a minute or more.




A
A'
F
F'
G
G'
D'
D
E
E'
B'
B
C'
C




H'
H
E
E'
D
C
C'
G'
G
F'
F
A'
A
B
B'

(2) Plotting points[a]

The problem of making a stereogram can be outlined as follows: Assume that the eyes (R and L in Figure 5.9) are at some definite position relative to the sheet of paper on which the stereogram is to be drawn. Suppose we want to represent a three-dimensional object such as a cube. We must first represent important points of the object (the vertices of the cube, for example) and then determine how to locate the red and blue images of each such point.

Figure 5.12 represents this situation. A cube is placed on a sheet of graph paper on which we have drawn an X - Y coordinate system. If we take the Z - axis to be perpendicular to the plane of the paper, then we can locate each vertex of the cube by its coordinates (x, y, z). Figure 5.12a shows a top view of the problem we want to solve: the observer sees the points (vertices of the cube) against the background of the graph paper. Each point P on the cube corresponds to a red (x) and a blue (o) image on the paper. Figure 5.12b shows a side view of the same situation. We must now find a way to locate those image points for a given point P = (x, y, z).

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[a] The calculations in this activity and in the applications are based on D. W. Stover, "Stereograms" (Houghton-Mifflin, 1966).

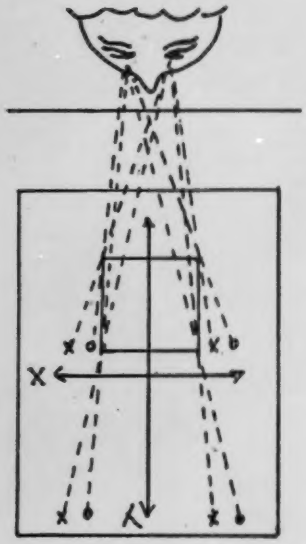


Figure 5.12a

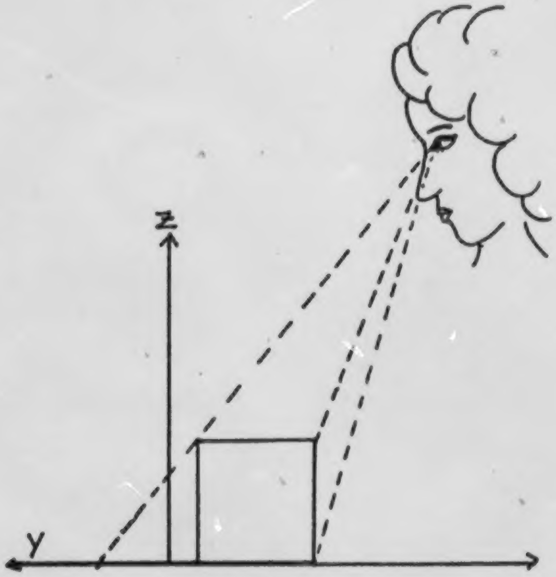


Figure 5.12b

In the development which follows, we will always make reference to a standard graph paper ruled 5 squares to 2.5 cm. We also assume the paper is 21.5 cm by 28 cm. The same scale must be used on the Z-axis: 1 unit = 0.5 cm. If you use some other scale for your graph paper you will have to modify the following derivations accordingly.

For the average person seated at a desk with the stereogram on the desk in front (as in Figure 5.12a), a reasonable set of coordinates are L = (-6, -61, 72) for the left eye and R = (6, -61, 72) for the right eye. These are determined by using the average separation of the pupils of the eyes (6 cm) and the scale of the graph paper (2.5 cm = 5 units). This gives the X-coordinates of -6 and 6. The other coordinates are estimated from the average height of the eyes above the desk (for the Z-coordinate of 72) and a reasonable viewing distance from the paper (for the Y-coordinate of -61, since distances between the X-axis and the observer are measured as negative on the Y-axis). Be sure you understand the meaning of these coordinates before you proceed.

Coordinates of left eye: L = (-6, -61, 72)

Coordinates of right eye: R = (6, -61, 72)

Since the graph paper in 21.5 cm by 28 cm, its size imposes some limitations. We want to place an object in front of us so that the red and blue image points fall on the graph paper. For our graph paper, it turns out that an object will have its image on the desired space if the object itself lies within the following range:

X-coordinate between -10 and 10: $(-10 \leq X \leq 10)$

Y-coordinate between -23 and -3: $(-23 \leq Y \leq -3)$

Z-coordinate between 0 and 22: $(0 \leq Z \leq 22)$

We are going to define a function which maps points in the object space onto points on the paper. Suppose we consider only points with integer coordinates where

$$-10 \leq X \leq 10$$

$$-23 \leq Y \leq -3$$

$$0 \leq Z \leq 22$$

How many such points would be in the domain of our function?

Review what we mean by distance in the three-dimensional coordinate system. Specifically, for two points $P_1 = (X_1, Y_1, Z_1)$ and $P_2 = (X_2, Y_2, Z_2)$

--What is the X-distance between the points?

--What is the Y-distance between the points?

--What is the Z-distance between the points?

--What is the distance between the points?

Let us now take a point $P = (X, Y, Z)$ on the object which we want to map onto the paper. We will first concern ourselves with locating P_L, the image of P seen by the left eye (L)--i.e., the red image. We will denote the coordinate of P_L as (X', Y', Z'). But since P_L lies on the graph paper, its Z-coordinate will be zero. Hence, $P_L = (X', Y', 0)$. Further, since we are viewing P_L with the left eye, we have coordinates of $L = (-6, -61, 72)$. Figure 5.13 illustrates this situation from a side view. It also shows the Y and Z distances between L and P and between L and P_L. (Remember that because of the limitations of our paper, $Y \geq -61$ and $Z \leq 72$. You must convince yourself that we do not need absolute value signs for the Y and Z distances in this case.)

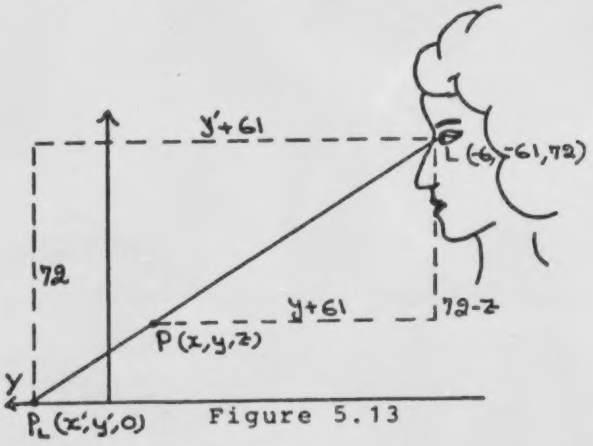


Figure 5.13

Notice that the horizontal and vertical lines in Figure 5.13 give us similar triangles. (Show this. Also list the corresponding parts.) Using the proportions from the similar triangles we get

$$\frac{Y' + 61}{Y + 61} = \frac{72}{72 - Z} \qquad (1)$$

$$\text{thus} \qquad Y' = \frac{72}{72 - Z} \cdot (Y + 61) - 61 \qquad (2)$$

Equation 2 gives the Y-coordinate of point P_L in terms of the Y and Z coordinates of P.

Now consider Figure 5.14, a top view of the same situation. In the figure the X and Y distance between L and P and between L and P_L are shown. Note that this time the X distances must include the absolute value symbols since the allowable range of X ($-10 \leq x \leq 10$) includes the possibility that P may be to the left of L. Once again the distances in the directions of the coordinate axes give similar triangles. (Where? What are the corresponding parts?) Using the proportions from these triangles we get:

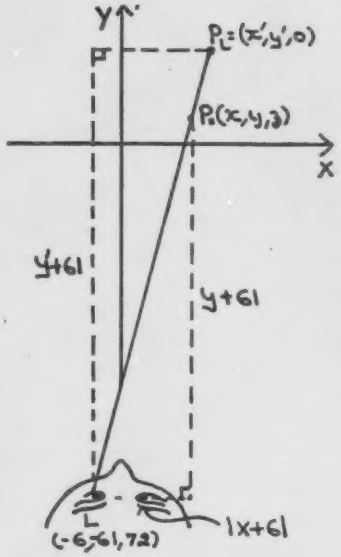


Figure 5.14

$$\frac{|X' + 6|}{|X + 6|} \cong \frac{Y' + 61}{Y + 61} \qquad (3)$$

By combining equations 1 and 3 we can write:

$$\frac{|X' + 6|}{|X + 6|} = \frac{72}{72 - Z} \qquad (4)$$

Now look again at Figure 5.14. We noted earlier that the X-distances must include the absolute value symbols here, because point P could be to the left of L. However, it should be clear from Figure 5.14 that if P is to the left of L, then P_L would also have to be to the left of L. (Check this by picking a point P* to the left of L and drawing the sight line $\overline{P^*L}$.) Since this is so, if (X + 6) is a negative number, so, too, will (X' + 6) be negative. In other words, the ratio on the left in equation 4 can never be negative. Hence, we can write:

$$\frac{X' + 6}{X + 6} = \frac{72}{72 - Z} \qquad (5)$$

$$\text{thus} \quad X' = \frac{72}{72 - Z}\,(X + 6) - 6 \qquad (6)$$

Equation 6 given us the desired transformation for locating X' once we know X. Now we have in equations 2 and 6 the necessary transformations for finding P_L, the (red) image of P seen by the left eye (L). What about P_R, the (blue) image of P seen by the right eye, R?

We could repeat the above for the right eye, but we do not need to do so. Look again at Figure 5.13 and the geometry of the problem for the right eye. Since R has the same Y and Z coordinates as L, the image $P_R = (X", Y", Z")$ will have the same Y and Z coordinates as P_L--i.e., Y" = Y' and Z" = 0. Only the X" coordinate is different, so we can simplify the problem by determining the separation (S) between X' and X". Then we can plot P_L and move a distance S to locate P_R.

To do this, consider Figure 5.15 which shows a top view of the right eye observation. We denote P_R by the coordinates (X", Y", 0). By the same argument that we used to determine X', we can show that

$$X" = \frac{72}{72 - Z} (X - 6) + 6 \qquad (7)$$

Thus the distance between X" and X' is given by the difference between equations 7 and 6 or

$$S = \left[\frac{72}{72 - Z} (X - 6) + 6\right] - \left[\frac{72}{72 - Z} (X + 6) - 6\right]$$

$$= \frac{72}{72 - Z} \left[(X - 6) - (X + 6)\right] + 6 - (-6)$$

$$= \frac{72}{72 - Z} (-12) + 12$$

$$= \frac{72 (-12)}{72 - Z} + \frac{12 (72 \div Z)}{(72-Z)}$$

$$= \frac{-12Z}{72 - Z}$$

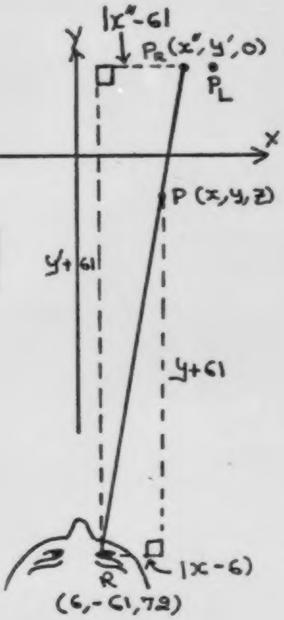


Figure 5.15

Hence, $S = \dfrac{-12Z}{72 - Z}$ (8)

In the above derivation we have arranged the difference in such a way that S turns out to be negative, since we move left (i.e., in the negative direction). We now have the desired transformations for finding the images of P:

$$X' = \frac{72}{72 - Z}(X + 6) - 6 \qquad (6)$$

$$Y' = \frac{72}{72 - Z}(Y + 61) - 61 \qquad (2)$$

$$S = \frac{-12\,Z}{72 - Z} \qquad (8)$$

(3) Applications

Find the P_L and P_R images of the following points and plot them on the stereogram:

(a) A = (-8, -10, 4) D = (7, -17, 4)
B = (-4, -6, 4) P = (-6, -8, 16)
C = (3, -21, 4) Q = (5, -19, 16)

(b) Draw red and blue segments which correspond to the images of the following:

$\overline{AB}$, $\overline{AC}$, $\overline{BD}$, $\overline{CD}$, $\overline{PA}$, $\overline{PB}$, $\overline{QC}$, $\overline{QD}$, $\overline{PQ}$

(A, B, C, D, P, Q are the points in (a) above.) The results should be a stereogram of a triangle prism.

(c) Make a stereogram of an object of your choice. Include your calculations for locating the points.

Teaching Notes

The field of geometrical optics is a rich source of activities involving spatial relationships, and the investigations in this chapter are just some of the activities which could be developed. The study of reflections (Investigations 5.1 and 5.2) usually is conducted using plane mirrors. However, the MIRA offers the additional advantage of producing an image which can be traced, so it is preferred over mirrors. The MIRA can be purchased commercially or it can be made from sheets of plexiglass purchased at a hardware store. The ends can be of any material. A light balsa wood works well and is easy to fashion into an endpiece. The important thing in making your own MIRA is to bevel the lower edge so that the mirror edge is half the thickness of the plexiglass. This can be done with an ordinary workshop file.

Investigation 5.1 should lead to generalizations about the size, shape, location, and position of images in a plane mirror. These generalizations provide the background for defining the three basic rigid transformations: translations (slides), reflections (flips), and rotations (turns). Students should determine the necessary and sufficient parameters to describe completely each of these rigid transformations:

A translation is determined by specifying the distance and the direction of the slide (i.e., a translation vector).

A reflection is determined by specifying the line of reflection (i.e., the mirror line).

A rotation is determined by specifying the center of rotation and the angle through which the rotation occurs.

We can define the product of two transformations, p * q, as the transformation q followed by the transformation p. Thus, we write p * q = r if r is a single transformation which produces the same outcome as q followed by p. (Perhaps the most familiar example of this is vector addition, where the vector quantities represent translations as in the case of displacements or forces.) Further, every rigid transformation can be decomposed into a product of translations, reflections and rotations, and we can extend this result to a definition of congruence: two figures are congruent if there is a rigid transformation (or product of transformations) by which one figure can be made to correspond to the other. In illustrating products of transformations it is necessary to trace the intermediate images which in turn are reflected in subsequent transformations.

In Investigation 5.2, the MIRA again replaces a plane mirror in illustrating the law of reflection: the angle of incidence equals the angle of reflection. Be sure that pupils include the special case of light perpendicular to the plane of the mirror (i.e., angle of incidence = 0^{o}).

For the multiple reflections in Investigation 5.3, mirrors are used. An inexpensive source of mirrors are the mirror tiles sold in discount, hardware, or home-care stores. These can be cut to convenient sizes. It is a good idea to glue the mirror to a block of wood so the mirror will stand perpendicular to the table. Also, masking tape can be used to hinge two blocks together.

Students should derive a relationship between the dihedral angle between the mirrors and the regular polygons formed by the multiple reflections of the line segment in the first part of this activity. When using three or more mirrors to form dihedral kaliedoscopes, pupils can generate artistic patterns which cover the plane. This latter activity suggests many extensions and enrichment projects. One such extension is to photograph the kaliedoscopic patterns. Another is to investigate the tessellations in various works of art, especially the works of M. C. Escher. A more quantitative task is to show how the law of reflection accounts for the multiple reflections which are observed.

Investigation 5.4 introduces the principle of refraction. The activity uses small plastic boxes filled with water, but glass prisms also can be used if they are available. Closely related to the refraction phenomenon is the dispersion of white light into the spectral colors. This effect is reasonably easy to observe and to discuss in a qualitative way, but good quantitative analysis would require more sophisticated equipment so it is not considered here. However, extensions of this investigation could include a study of the spectra of various elements.

The simple pin-hole activity of Investigation 5.5 introduces other relationships which yield measurable ratios. The magnification $\frac{s_i}{s_o}$ should equal the ratio $\frac{d_i}{d_o}$, which is easily verified using similar triangles. In Investigation 5.6 a convex lens is used to focus the light. As the relative positions of the candle and screen vary, students should observe the following patterns:

(1) As the candle moves in from $d_o = \infty$ to $d_o = 2f$, an inverted image is formed on the screen starting at $d_i = f$ and moving toward $d_i = 2f$. As the image moves out from f to 2f, its size increases. When $d_o = d_i = 2f$, object and image should be the same size.

(2) As the candle moves in from $d_o = 2f$ to $d_o = f$, the inverted image moves out from $d_i = 2f$ toward $d_i = \infty$. As it moves out, s_i increases until finally the image vanishes when $d_o = f$.

The measurements made in part three of Investigation 5.6 should be sufficient to suggest the basic lens formula: $\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$. As in the case of the pinhole, the proportion $\frac{s_i}{s_o} = \frac{d_i}{d_o}$ also holds for lenses.

The fourth part of this activity concerns the case of an object placed inside the focal length of the convex lens ($d_o < f$). For this case, object and image appear on the same side of the lens and the image is erect, not inverted. As the object approaches the lens, the image moves in from infinity and decreases in size. In the limit, the object and the image meet at the lens where they are of equal size.

The phenomenon of image formation lends itself to numerous geometric investigations in which the path of the light is traced and images are located and described. Investigation 5.7 illustrates these situations for spherical mirrors. Similar problems could be developed with reference to convex and concave lenses, and Investigation 5.6 is natural to motivate such activities. The questions one might ask would be analogous to those of Investigation 5.7. Note, too, that the generalizations discussed relative to Investigation 5.6 are similar to the conclusion one expects from Investigation 5.7. Another situation suggested by the spherical mirrors is to study the images formed by parabolic reflectors, which produce sharper images and more well-defined beams because the geometry of a parabolic reflector eliminates the distortion (called spherical aberration) which results from light reflected near the extremities of a spherical mirror.

Finally, Investigation 5.8 shows an application of basic principles of optics in the creation of three-dimensional illusions. The pamphlet "Stereograms" by D. W. Stover (Houghton-Mifflin, 1966) gives an extended discussion of the principles from which the transformations are derived. It also contains tables of transformations which are extremely useful in constructing original stereograms. If students have difficulty "seeing" the stereograms, two possible causes are (a) they have drawn the lines too dark or (b) they did not stare at the paper long enough.

All of the investigations in this chapter are based on spatial relationships associated with various aspects of the field of optics. There are, however, numerous other sources of activities which introduce spatial relationships. Some of these are suggested below.

Other Applications

1. Reflection and refraction occur for wave phenomena other than light waves. For example, a ripple tank can be used to study these properties of water waves.
2. Interference results when waves spread out after passing through a narrow opening. The study of interference

patterns in light, sound, or water waves leads to other interesting spatial relationships.

3. Billiard balls also obey the law of reflection. Their behavior can be studied on conventional rectangular tables or on tables shaped like equilateral triangles, rhombi, or ellipses.
4. A vibrating string produces standing wave patterns which can be studied as the frequency of the vibration changes.
5. Chicago's Museum of Science and Industry and the United States Capitol in Washington, D.C., both contain rooms in which sound is reflected to a focus point. A whisper at one focus in the room is clearly audible at the other. What properties of the geometry of these rooms would account for these effects?
6. Investigate the combinations of lenses and/or mirrors used on the following instruments:

    telescope
    microscope
    periscope
    kaleidoscope
    stereoscope

7. A tree casts a shadow which varies in length and position at different hours of the day. Investigate the relationship between the size, shape, and position of an object and its shadow relative to different

positions of the light source. (This investigation can suggest many topics of projective geometry.)

8. During an eclipse, the center of the shadow is totally dark (the umbra) while the surrounding part is partially illuminated (the penumbra). What is the reason for this effect?

9. When a magnet is brought near iron filings, the filings align themselves in definite patterns. These patterns can be studied to describe properties of the magnetic field.

10. The atoms in a crystal are in a well-defined pattern. X-rays focused on the crystals are diffracted and produce patterns which reveal the hidden structure of the substance.

11. Minerals exhibit a property called cleavage: the tendency to break along smooth planes. The cleavage patterns of various materials reveal many different polyhedra.

12. Symmetry and patterns are abundant in nature. The following are some sources worth investigating:

    flower shapes
    the arrangements of leaves on a branch
    spider webs
    pine cones
    sea shells

beehives

soap bubbles

13. During the 1950s, movie companies produced three-dimensional ("3-D") films which featured people, animals, monsters, and objects which "jumped off the screen" at the audience. Investigate how these films used the principles of stereograms to create their special effects. (Note: Instead of red and blue light directed at one eye or the other, 3-D movies usually used polarized light and the audience wore "glasses" with polaroid lenses.)

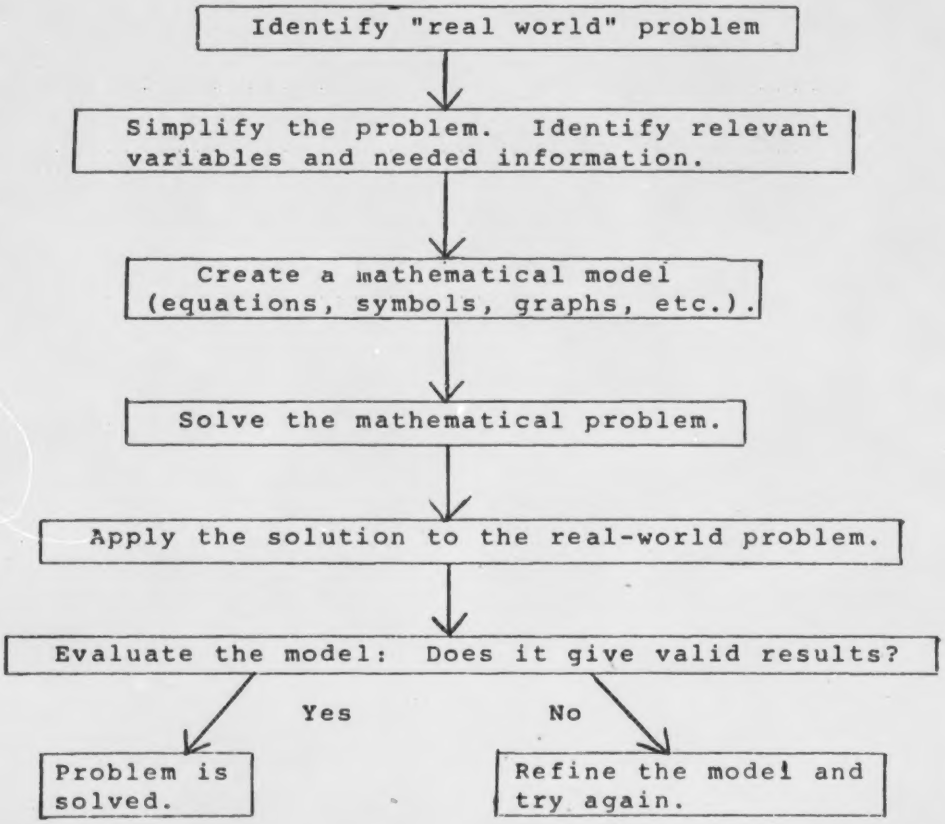
14. A sand pendulum is constructed using a funnel filled with sand as the pendulum bob. Investigate the patterns formed as the sand runs out from the swinging pendulum. Allow the pendulum to swing in different arcs (i.e., do not restrict its motion to one plane).

15. A child's toy called a Spirograph (Kenner Corp.) uses gears of varying ratios to generate numerous patterns. The spatial relationships involved in many of these also can be used to illustrate aspects of Ptolmaic astronomy such as the theory of epicycles of the planets.

Chapter 6

Modeling, Predicting, and Decision Making

Mathematics is created out of our need to solve problems which can arise from many sources. Usually these sources have some basis in reality, although we often step into the world of mathematics for their solution. In solving real-world problems, both the scientist and the mathematician frequently construct models. The modeling process can be outlined as follows:


Identify "real world" problem
Simplify the problem. Identify relevant variables and needed information.
Create a mathematical model (equations, symbols, graphs, etc.).
Solve the mathematical problem.
Apply the solution to the real-world problem.
Evaluate the model: Does it give valid results?
Yes
No
Problem is solved.
Refine the model and try again.

To be useful our model must do two things: it must explain known phenomena and it must enable us to predict yet unknown outcomes. If it cannot do these things, the model must be adjusted or -- sometimes, though usually reluctantly -- it must be replaced.

Many excellent case studies of the evolution of a model are found in the history of science. These include developments of models of the solar system, the atom, the nucleus, light, electromagnetism, DNA, the universe, biological evolution, radioactivity, and more. Each could be studied to determine the assumptions and abstractions upon which the model was based; the observed phenomena it sought to explain; the aspects of the real problem which were adequately accounted for; the discrepant events which the model could not explain; the modifications of the model proposed to account for such discrepancies; and the adequacy of the modified model.

For example, the ancient Greek model of the universe placed the earth at the center with the sun and planets in circular orbits around it. While this model could explain many observations, such as the daily motion of the sun from its rising on the eastern horizon to its setting on the western horizon, it failed to account for the "wandering" or "retrograde motion" of the planets. This latter term describes the observed behavior of the planets which appear at certain times to move eastward among the stars, then to reverse and for several months to move westward. In an attempt to explain this phenomenon, Ptolemy's model of the universe preserved the earth as the center of the system but proposed the modification of epicycles. In this model, each planet moved at a uniform rate on the circumference of a small circle (the epicycle) while the center of the epicycle moved in a large circle around the earth.

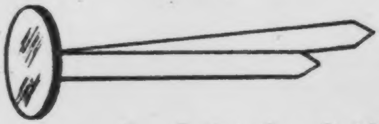
While this modification of the model successfully accounted for one previously unexplained observation (retrograde motion), other discrepancies remained. Further modifications were necessary such as displacing the Earth from the geometric center of the motions. Finally, the complexities of the Ptolemaic model led to its replacement, but this did not occur until some 1500 years after Ptolemy published his theory. Even when proposing the model which ultimately replaced Ptolemy's, Copernicus himself wrote of his attempts to improve on Ptolemy's model, not to overthrow it. Only when he was convinced that his sun-centered theory led to a simpler system did Copernicus reluctantly change the model.

The models we use may employ equations, graphs, diagrams, symbols, or physical materials. Investigation 2.17 presented a model of radioactive decay using physical materials (dice). Many of the other activities in this book used graphs or diagrams to represent or model relationships. In this chapter we will explore other models which are useful in helping us understand and explain observed phenomena, predict new outcomes, and make decisions based on those predictions.

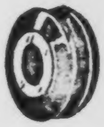
Investigations

Investigation 6.1

A simple model of chemical composition is found in the Introductory Physical Science (IPS) course (Educational Development Center, 1967). This model uses paper fasteners and small rubber rings to represent "atoms" of two "elements" called Fs and R. Different combinations of fasteners and rings also can be fit together to represent "compounds."



An "atom" of the "element" Fs.



An "atom" of the "element" R.

Figure 6.1

A "molecule" of the "compound" FsR.

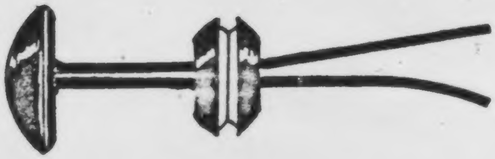


Figure 6.2

Figure 6.2 shows a "molecule" of the "compound" FsR.

(1) Assuming that you have a large supply of both Fs and R and assuming that all the fasteners are alike and all the rings are alike, show how this model can be used to illustrate the following principles:

(a) All the atoms of a particular element are identical.

(b) The mass of a substance is proportional to the amount of that substance which is present.

(c) Atoms can combine in different ways to form different compounds. For example, illustrate the following:

FsR Fs_2R FsR_3 Fs_2R_3

FsR_2 Fs_2R_2 $2FsR$ $3FsR_2$

(d) In a given compound, the ratio of the mass of Fs to the mass of R is always the same (Law of constant proportion).

(e) When Fs and R combine to form different compounds, the masses of R which combine with a fixed mass of Fs are in the ratio of small whole numbers (Law of multiple proportions).

(f) In a mixture of Fs and R, the elements can occur in any proportion.

(g) When elements combine to form compounds or mixtures or when compounds or mixtures are decomposed into elements, matter obeys the law of conservation of mass.

(2) Evaluate this model for its ability to illustrate known principles of chemistry. Also indicate weaknesses in the model.

(3) Suggest other objects besides fasteners and rings which might be used to create a similar model.

Investigation 6.2

Another physical model also found in the IPS course uses a mechanical device to represent the kinetic theory of gases. In this device, a plastic cylinder is fitted with two movable discs. The lower disc is attached to a small motor which causes it to vibrate up and down. Small steel spheres contained between the two discs bounce when the lower disc vibrates, and the impact of the bouncing spheres on the upper disc causes that disc to rise.

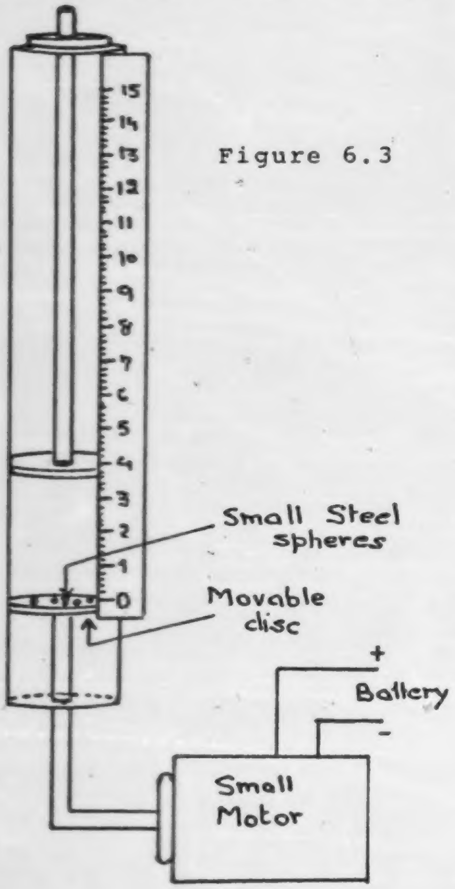


Figure 6.3

The spheres in this model represent the molecules of a gas. The speed of the motor determines the speed of the spheres as the temperature of a real gas determines the speed of its molecules. Small rubber rings such as those in the previous activity can be placed on the upper disc to exert additional pressure against the gas, thus tending to compress it. Use this model to investigate the behavior of gases by answering questions such as the following:

(1) How does the speed of the molecules affect the volume of the gas?

(2) How does the number of molecules in the cylinder affect the volume of the gas if the speed (temperature) of the molecules is constant?

(3) How does the pressure exerted against the gas affect the volume if the speed (temperature) is constant?

(4) If the pressure against the gas is doubled, what must be done to the speed (temperature) in order to maintain a constant volume in the gas?

(5) If the number of molecules in the gas is doubled, what must be done to the pressure in order to maintain the gas at the same temperature and volume?

(6) How useful is this mechanical model for explaining the behavior of a gas?

(7) What are the limitations of this model?

Investigation 6.3

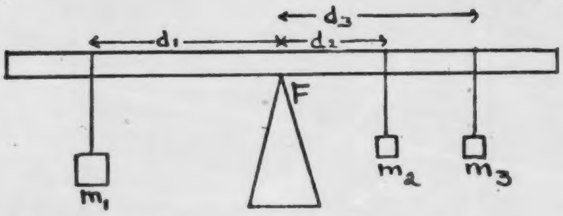


Figure 6.4

Figure 6.4 shows a lever supported at a point F called the fulcrum. Several weights are hung from the lever as shown. The product $m_i d_i$ for a mass m_i hung at a distance d_i from the fulcrum is known as the moment of force about F, a measure of the turning effect of that force acting on the lever. In order for the lever to balance, the sum of the clockwise moments on one side of the fulcrum must equal the sum of the counterclockwise moments on the other side. For the case illustrated in Figure 6.4, balance is achieved when

$$m_1 d_1 = m_2 d_2 + m_3 d_3.$$

We can use a lever such as this as a physical model of the arithmetic mean of several numbers. For the model, use a meter stick for the lever and a number of equal weights such as steel washers for the masses. Suppose, for example, you want to find the mean of the numbers 33, 48, 56, 85, and 92. Hang one washer from each of the following points: 33 cm, 48 cm, 56 cm, 85 cm, 92 cm. Find the point where the meter stick can be made to balance. This point (62.8 cm) is the mean of the given numbers. Why? Explain how the lever is a model for the mean. Illustrate a number of other sets of numbers and find the mean of each. Is it essential to use identical weights in this model? Why or why not? Under what conditions might you use different weights? What would you represent if you did that?

Investigation 6.4

A class of popular mathematics problems is illustrated by the following example:

> A scientist in a poorly equipped lab has only two containers for water: a five-liter one and a three-liter one. How can she measure exactly one liter? (Note: The containers are completely unmarked. It is only possible to fill or empty them. It is not possible to measure fractions of the total volumes directly.)

Before you proceed, solve the problem above. Record your solution in a chart like the following:

Step Number	Contents of 5-liter jar	Contents of 3-liter jar	Description of activity
0	0	0	
1			
2			
3			
etc.			

In the column at the right write a description of what has happened in that step. (For example, "Empty small jar into large jar.")

Martin Gardner[1] proposed a mathematical model for problems of this type. The model is based on a hypothetical billiard table shaped like a parallelogram. The lengths of the sides are determined by the sizes of the containers in the problem. Balls on this table obey the usual physical laws of reflection--i.e., the angle of incidence is equal to the angle of reflection. We diagram the table for our problem below using isometric dot paper.

---

[1] "Bouncing Balls in Polygons and Polyhedrons," in the Sixth Book of Mathematical Games from Scientific American, Scribner, 1971.

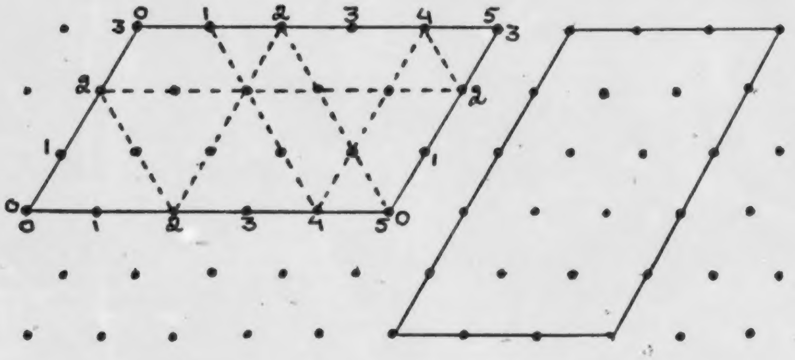


Figure 6.5

In the above (Figure 6.5) model, our ball leaves from (0,0) along the horizontal axis to (5,0). At this point (a), the ball is reflected to (b) which has coordinates (2,3). In similar manner the ball travels from b to c to d . . . to h.

(a) For each reflection point a through h, name the coordinates of the point.

(b) Interpret the meaning of the above in terms of filling and emptying containers.

Another way to represent the solution is as follows:

5-liter container	0	5	2	2	0	5	4	4	1
3-liter container	0	0	3 ↓	0	2	2	3 ↓	0	3

(c) Explain the meaning of the above representation.

(d) What would have been the outcome if we had started our ball from the origin to the 3-liter coordinate instead of starting with the 5-liter coordinate? Use the table at the right in Figure 6.5 to solve this problem. Give the coordinates of each reflection and interpret them in terms of filling and emptying.

(e) Using the billiard table model, determine which volumes in addition to one liter can be obtained using the 5- and 3-liter containers.

--How are these shown by the model?

--List the steps in obtaining each of the possible volumes.

--Is there only one way to obtain a given volume? If there are several ways, how can the model tell you which way is the most efficient (i.e., has the fewest number of steps)?

(f) Now let us try to extend the model to new situations. A natural question involves billiard tables of other dimensions (or, equivalently, containers of different volumes). Using the isometric dot paper, model each of the following cases. For each case determine all of the volumes which can be measured using the two containers.

3 by 5
3 by 7
5 by 7
2 by 5
5 by 5
5 by 6
4 by 6
4 by 5
3 by 6
5 by 8
3 by 9
6 by 9

(g) Is it always possible to measure any desired volume? If not, under what conditions is a volume impossible?

(h) Generalize your findings from above: For given volumes A and B, which volumes can be measured?

(i) Now let's extend the model once more to a more complex problem. In the above cases we assumed that there was an unlimited supply of water and that we could get more water or throw water away at will. Now assume instead that you have an 8-liter container of valuable liquid (gasoline, for example). You also have empty containers of 3 and 5 liters (one of each size). How can you divide the 8 liters of gasoline into two equal parts? Solve the problem before you go further.

Figure 6.6 shows the billiard model for the 3- and 5-liter containers as before. We have added line m parallel to the long diagonal of the billiard parallelogram. The coordinates shown on line m are determined by taking the intersection of m with the grid lines which are parallel to line n in the diagram. The length of m is equal to the length of the long diagonal.

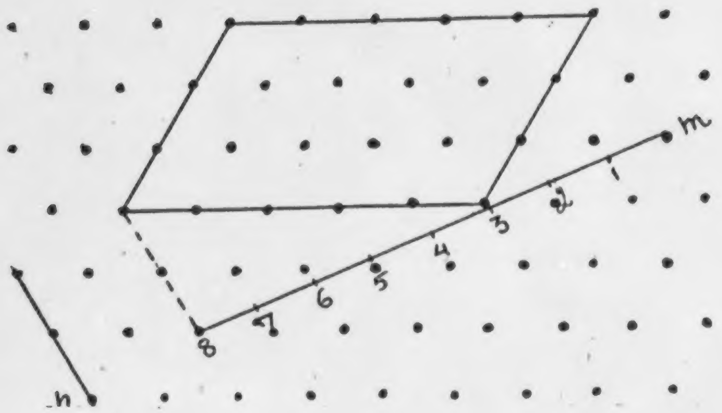


Figure 6.6

(j) Show how the coordinates shown on m are related to the coordinates of the points in the billiard region. (Hint: Consider the lines which were used to determine the points on m. The dotted line in Figure 6.6 is one such line.) Explain the above relationship in terms of the volumes in the three containers.

(k) Use the model to solve the problem given on the previous page. How do you know when you have the solution? Is the solution unique? (i.e., can you divide the 8 liters in more than one way?)

(l) Using the above model, determine all the possible ways in which you can divide the 8 liters using the 3- and 5-liter containers.

(m) Return to the different combinations which you investigated in (h). Show that, for the cases you have checked, if A and B are two containers such that $A < B$ and A and B meet the necessary conditions which enable you to measure one liter, then if we have a third container C which is full and for which $c \geq A + B$, we can measure any unit volume up to B. (For example, if $A = 15$ and $B = 16$ and $c \geq 31$, then we can measure any amount from 1 through 16.)

(n) Try to account for the cases in h where you could not measure 1 liter. In these cases, if there is a third container C (as described above) where $C \geq A + B$, tell which amounts you can measure using A and B.

(o) Consider now cases where the container C is less than the sum of A and B. See if you can modify the billiard model to allow you to solve the problem of dividing the contents of C. (Here is an example of how a model no longer seems adequate and must be modified to account for new phenomena.)

(p) (The following challenging problem is found in the puzzles of Sam Lloyd.)

Some soldiers manage to capture a 10-gallon keg of beer. They naturally sampled part of it, making use of a 3-gallon and a 5-gallon container. The rest of the beer was carried back to camp in three equal portions--one in each of the containers and one in the keg. How much did they drink and how did they measure the remainder into three equal (non-zero) parts?

(The best solution is the one with the fewest steps for the entire process. Each step, including the drinking operation, involves an integral number of gallons. It goes without saying that the soldiers would never think of wasting any beer by throwing it out.)

Investigation 6.5

Simple electric circuits provide a model for studying algebraic systems. The two basic cases are illustrated in Figure 6.7. In each circuit A and B represent switches which can be either open (0) or closed (1). A battery supplies power and we observe the light bulb in the circuit to determine if it is on (1) or off (0).

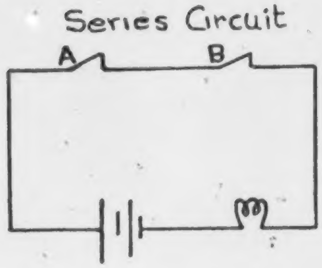


Figure 6.7a

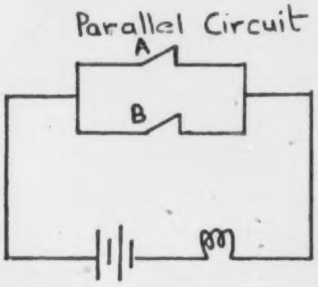


Figure 6.7b

Since the bulb can be lit only if there is a closed path through which the electricity can flow, we can determine the condition of the light bulb for all possible conditions of the switches.

(A) Complete the tables below to show whether the bulb is on (1) or off (0).

Series Circuit

A	B	Bulb $(A \cdot B)$
1	1	
1	0	
0	1	
0	0	

Parallel Circuit

A	B	Bulb $(A+B)$
1	1	
1	0	
0	1	
0	0	

(B) We will use the symbol "." to represent electricity flowing in the series circuit and the symbol "+" to represent electricity flowing in the parallel circuit. Verify each of the following generalizations about the circuits:

$$A + B = B + A$$

$$A \cdot B = B \cdot A$$

$$0 \cdot X = 0$$

$$1 \cdot X = X$$

$$0 + X = X$$

$$1 + X = 1$$

From the observations you made so far, do there appear to be identity elements for $\cdot$ and for $+$?

(C) In Figure 6.8 a third switch has been introduced. Complete the tables and use the information in them to establish the following associative properties:

$$(A \cdot B) \cdot C = A \cdot (B \cdot C)$$

$$(A + B) + C = A + (B + C)$$

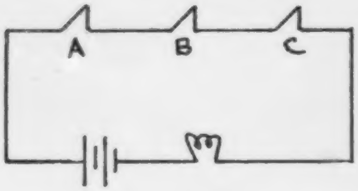


Figure 6.8a

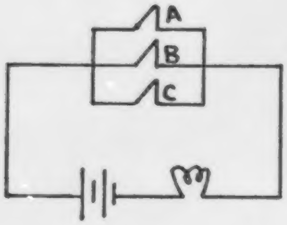


Figure 6.8b

A	B	C	Bulb (A·B·C)
1	1	1	
1	1	0	
0	1	1	
1	0	1	
0	0	1	
1	0	0	
0	1	0	
0	0	0	

A	B	C	Bulb (A+B+C)
1	1	1	
1	1	0	
0	1	1	
1	0	1	
0	0	1	
1	0	0	
0	1	0	
0	0	0	

(D) Two more complex circuits are shown in Figure 6.9.

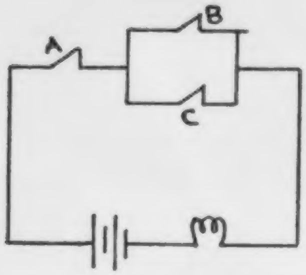


Figure 6.9a

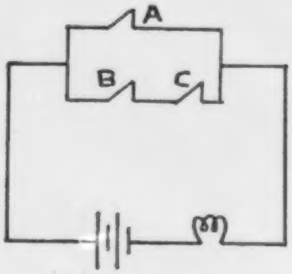


Figure 6.9b

Which of the circuits represents each of the following:

$A + (B \cdot C)$

$A \cdot (B+C)$

Use the circuits in Figure 6.9 to illustrate the following two distributive properties:

$A + (B \cdot C) = (A+B) \cdot (A+C)$

$A \cdot (B+C) = (A \cdot B) + (A \cdot C)$

Which of the above is equivalent to the distributive property of the real numbers?

(E) Now suppose we introduce a double throw switch as illustrated in Figure 6.10. This switch has the property of being able to close in either direction (A or A'); however, if A is closed, then A' must be open and vice versa.

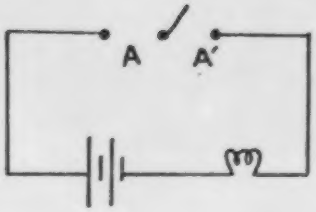


Figure 6.10a

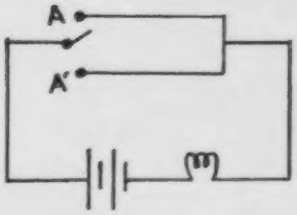
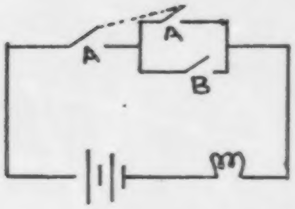


Figure 6.10b

From Figure 6.10a, determine the value of A•A'.

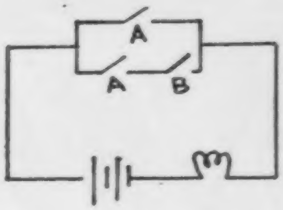
From Figure 6.10b, determine the value of A+A'.

(F) If we use a double-pole switch, we can create the situations in Figure 6.11. With this switch, both poles must either be open or closed, so we will use the same letter to represent each. Use the diagrams to determine the value of the given expressions.



A • (A+B) =

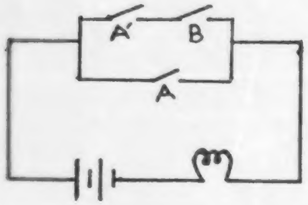
Figure 6.11a



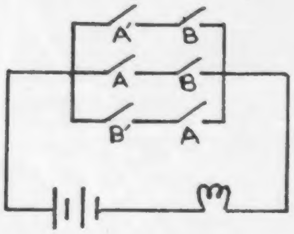
A + (A•B) =

Figure 6.11b

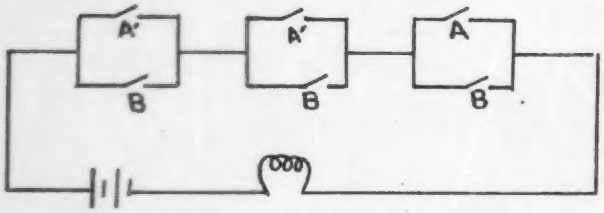
(G) Using the principles derived so far, simplify the following expressions:



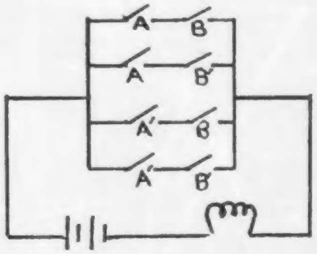
$A' \cdot B + A =$



$A' \cdot B + A \cdot B + B' \cdot A =$



$(A'+B') \cdot (A'+B) \cdot (A+B) =$



$A \cdot B + A \cdot B' + A' \cdot B + A' \cdot B' =$

Investigation 6.6

A problem encountered by groups like the Department of Natural Resources is to estimate populations such as fish in a lake or deer in a region. This task usually is accomplished using a mathematical model which relies on probabilities. You can investigate this model, known as the capture-recapture method, using simple equipment.

Select a large quantity of some convenient item, such as a dish of dried beans, to represent the population of interest. "Capture" a sample of the population by scooping some of the beans in your hand or in a small container. "Tag" the captured sample by marking them with a felt pen or with food coloring. Count the number of tagged beans and then return them to the original population and thoroughly mix the entire quantity. Once again capture a small sample and note the proportion of tagged beans in this second sample. Estimate the size of the original population using the following proportion:

$$\frac{\text{Number tagged in Sample \#2}}{\text{Total number in Sample \#2}} = \frac{\text{Total number of Sample \#1}}{\text{Total number in population}}$$

(1) Let S_1 = size of Sample #1

S_2 = size of Sample #2

t = number of tagged items in Sample #2

P = size of population

What is the smallest possible value of P?

(2) List the assumptions which underlie this model. Suppose you applied the model to an actual situation such as estimating the number of fish in a lake. What assumptions would be implicit in the model? What variables could affect the outcome?

(3) Examine the model further by considering the following hypothetical case:

$S_1 = 10$

$S_2 = 12$

$t = 5$

What is the minimum value possible for P? What is the value of P suggested by the model? You should find that $P \geq 17$ since the least number of fish or wildlife in this population would be (10 + 12 - 5). Why? However, the value of P predicted by the model is 24.

(a) Suppose P=17 and 10 of these animals are tagged. Compute the probability that a random sample of 12 animals would consist of 5 tagged and 7 untagged ones.

(b) Now suppose P=24 and 10 are tagged. Again find the probability that a random sample of 12 would consist of 5 tagged and 7 untagged.

(c) Repeat the above for P=18 through P=23 and for P=25 through P=30. Arrange all of your probabilities in a chart. Which value of P appears most likely?

(4) Suggest some situations for which the capture-recapture method would be useful. What are some decisions which one might make based on the estimates in these situations? What are the limitations in each case?

Investigation 6.7

The previous investigation presented a model for estimating a population at a given point in time. One important reason for wanting to know the size of a population is to monitor the increase or decrease of a species. Many of today's endangered species might still be plentiful if humans had made decisions and adopted policies to protect or control them. In this investigation we will consider a hypothetical endangered species of animals about which we know the following to be true:

- The animals reach maturity at age 3.
- Of the females aged 3 or older, 88 percent give birth to one cub per year.
- In the population, 55 percent of the cubs are female and 45 percent are male.
- Only 42 percent of the cubs live to maturity.
- Of the animals who do live to maturity, 15 percent die each year from natural causes (disease, predators, starvation, freezing, drowning, etc.).
- An additional X percent of the mature herd is killed each year by humans.

Suppose the population this year is 10,000 mature animals. We wish to predict the population in ten years. We will do this first for the female animals using the above data. We will use the notation $N(F_k)$ to denote the number of mature females in year k (we take the current year to be zero), and we note the following:

(1) Of the number of mature females who begin year nine, 15 percent will die of natural causes and X percent will be killed. Thus, the number of females who begin year ten will be

$$(.85 - X') \times N(F_9)$$

where $X' = \frac{X}{100}$ (since X represents a percent).

(2) In year ten, some of the cubs born during year seven will reach maturity. The number of cubs born that year is $.88 \times N(F_7)$ and of these 55 percent are female. However, only 42 percent survive until year ten. Thus, the number of mature females added to the herd in year ten is

$$(.88 \times .55 \times .42) \times N(F_7)$$

$$\text{or} \quad .203 \times N(F_7)$$

(3) Combining (1) and (2) we get

$$N(F_{10}) = (.85 - X') \times N(F_9) + .203 \times N(F_7)$$

We see from the above equation that the number of mature females in any given year depends on the numbers of previous years and on the rate of "harvesting" (killing) by humans. That equation gives a model for population control.

(4) Of the 10,000 mature animals today, we expect 5,500 to be female. Suppose we want to maintain

the herd at 5,500 females for each of the next ten years. Then our population model tells us that

$$N = (.85 - X')\,N + .203N$$

$$\text{or} \quad 1 = (.85 - X') + .203$$

since $N(F_{10}) = N(F_9) = N = 5{,}500$.

What limit should be placed on X' to assure a steady-state population of females? That is, how many mature females per 100 should be harvested each year?

(5) Suppose the rate of harvesting females is 10 percent. How many mature females can we expect in ten years? How many mature females will there be if all harvesting is strictly forbidden?

(6) Develop the population model for the male animals, $N(M_k)$. Determine the harvesting rate which would assure a steady-state population of mature males. [Caution: The number of mature males added in year ten depends on $N(F_7)$, not on $N(M_7)$.]

(7) Look up similar data for some actual species such as buffalo, sperm whales, bald eagles, or another animal. Develop a model for population control of that species. If possible, compare your conclusions to the policies actually in effect for the given species.

Teaching Notes

Models present an efficient way of viewing problems. By representing only what is essential in as simple a way as possible, models enable us to conceptualize relationships, to predict outcomes of various actions and to make decisions based on those predictions. Often our ability to solve problems depends on our ability to conceive effective models.

It was suggested in the introduction to this chapter that the history of science is in many ways a history of models. It is highly recommended that students trace the evolution of at least one model as suggested in that earlier section. Such study should emphasize the nature of the model at successive times, the strengths and limitations of each modification, and the relationships between earlier models and those which replaced them.

Sometimes models are quite abstract and symbolic as in the case of present-day models of the atom and the nucleus. However, models also can be very concrete and simple. The fasteners and rings in Investigation 6.1 are an example of a simple model. Note that this model represents quite well the laws of conservation of mass and of definite and multiple proportions. On the other hand, note too that it suggests nothing about the actual mass, volume, or structure of atoms and molecules. It also fails to suggest anything about the molecular motion of substances, a limitation which serves to motivate the need for another model such as the mechanical one of Investigation 6.2.

It should not be too difficult to obtain this "gas machine" for classroom demonstration, since many schools' physical science laboratories have one. The mechanism can be powered by a small battery and is easily assembled in any classroom.

The balance beam model for the mean is easily modified to accommodate any range of numbers. It also can be used to illustrate the effect which extreme values can have on the mean. (Check, for example, several values clustered around 50 and one value very near to 100 or to 0.) Also, use the model to observe the difference between the mean and the median of the set of values.

The billiard table model for solving measuring problems is a good illustration of a model which appears to be quite far removed from the situation which it represents. The investigation leads to the conclusion that if one can measure a unit volume (one liter, in this case), then one can measure any volume. Further, the condition which assures that one can measure the unit volume is that the capacities of the two measuring containers must be relatively prime numbers. This model is particularly useful not only because it tells if the problem can be solved but also because it shows how to solve it.

In Investigation 6.5 the model is based on simple electrical circuits. It is most effective if the students actually wire the circuits or if one set of wired circuits is available for classroom demonstration. The mathematical system which results is, of course, a boolean algebra. The degree to which one might investigate the properties of this algebraic structure depends on the mathematical maturity of the students. It is worth noting at least the associative, commutative, and distributive properties, in particular the existence of not one but two distributive laws; the existence of identities and complements; and the equivalence of multiplication (series circuit) and addition (parallel circuit) to "AND" and "OR," respectively. It also can be interesting for the students to explore some applications of boolean algebra in computers or other devices.

Finally, the two models for estimating and controlling populations can be studied for their properties as models and then can be employed in studying some real population of interest to the class. The capture-recapture method is based on probabilities and the example given in section (3) of Investigation 6.6 illustrates this for one specific case. In that example, the student is asked to compute the probability that the value of P is some specified P* (where $P^* \geq 17$. The prediction that $P = \frac{s_1 s_2}{t}$ rests on the assumption that the animals which are caught are a random sample of the population and that the tagged animals occur in the same proportion both in the sample and in the population. Thus, if S_2 consists of 12 animals, 5 tagged and 7 untagged, the question becomes:

> If there are P* animals in the population and 10 of these are tagged, what is the probability that a random sample of 12 animals will consist of 5 tagged and 7 untagged ones?

Mathematically the answer to the above question is given by the expression:

$$p(P^*) = \frac{({}_{10}C_5) \times ({}_{P^*-10}C_7)}{({}_{P^*}C_{12})}.$$

That is, the probability that the population consists of P* animals is given by

$$\frac{\begin{pmatrix}\text{Number of ways to} \\ \text{get 5 tagged ani-} \\ \text{mals out of 10}\end{pmatrix} \times \begin{pmatrix}\text{Number of ways to get} \\ \text{7 untagged animals} \\ \text{out of } P^*-10\end{pmatrix}}{\begin{pmatrix}\text{Number of ways to} \\ \text{get 12 animals} \\ \text{out of } P^*\end{pmatrix}}$$

For the case of P=17, we must evaluate

$$p(17) = \frac{({}_{10}C_5) \times ({}_{7}C_7)}{({}_{17}C_{12})} = \frac{\left(\frac{10!}{5!\,5!}\right) \times \left(\frac{7!}{7!\,0!}\right)}{\left(\frac{17!}{12!\,5!}\right)}$$

For the example given, the probability of P=17 is very low (0.04), but as P increases the probability rises to a maximum (0.32) for P=24, then decreases again for $P > 24$. Hence, the model predicts the most likely value of P.

The model for estimating a population at a given time later can be combined with the model for predicting the growth or decline of a population as given in Investigation 6.7. Together, the models give a basis for a number of important applications to problems of contemporary significance. Some other areas for studying models are suggested below:

Other Applications

1. Trace the development of one or more of the models suggested in the introduction:

   Models of the solar system
   the atom
   the nucleus
   light
   electromagnetism
   DNA
   biological evolution
   radioactivity
   the origin of the universe
   the size and composition of the universe
   relativity
   quantum mechanics

2. In March 1980, Mount St. Helens, a volcano in Washington, erupted after being dormant for 123 years. Although scientists have not seen the inside of a volcano, they

are able to construct models of their interior structure. Investigate the nature of the information which enables geologists to construct models of volcanic behavior. Some of the questions which such models should attempt to answer are: How does the molten material (called Magma) move through the earth's crust? Why do some volcanoes erupt with explosions while others produce quiet lava flows? What causes the formation of volcanic cones, craters, and calderas? How do volcanoes influence the surrounding landscape? Why do volcanoes appear more frequently in certain regions of the earth than in others? Are there ways to predict when a volcano might erupt?

3. Like the questions above concerning volcanoes, many things of interest to geologists cannot be studied directly. Often this is because the geologic time scale spans millions or billions of years. Investigate the geologic models which scientists use to explain the following:

   the formation of the planet Earth
   the interior of the earth
   the formation, movement, and effects of glaciers
   the evolution of a stream or river
   the formation of mountains
   the erosion of the landscape due to water, wind, temperature change, etc.

4. An important model used by astronomers is the Hertzprung-Russell (H-R) diagram, a graph which classifies stars according to their spectral class (color and temperature) and their absolute magnitude. Look up the meaning of the following types of stars and show how they are represented on the H-R diagram:

    main sequence stars (this includes our sun)
    red giants
    supergiants
    white dwarfs
    population I and population II stars
    variable stars
    novae

5. Like the geologist, the astronomer also relies heavily on models. Investigate the nature of the models used to explain the following:

    the life cycle of a star
    the composition and behavior of comets
    the composition and structure of the sun
    sunspots and solar flares
    the classification of galaxies as "spiral," "elliptical" or "irregular"
    black holes

6. Maps are a type of model. Collect several examples of geographic, topographic, nautical, and political maps,

identify the features which are represented on each. In what ways are the maps alike and in what ways are they different? What adjustments must be made in order to represent the surface of the earth on a flat map?

7. The periodic table of the elements is another familiar model. Identify the ways in which this model represents known characteristics of the elements.
8. Some important social as well as scientific issues include energy consumption and production; disease control; the use and conservation of natural resources; weather prediction; land use and management; waste product disposal; and the environmental impact of various industries, construction projects, etc. Investigate the ways in which models are used in some aspects of these problems. How do models help in the decision-making process?
9. Life on a planet depends on many factors including the temperature, the composition of the atmosphere, and the availability of food and water. Investigate some of the models which contemporary astronomers have developed to predict the probability of planets in other solar systems with temperature and atmospheric conditions similar to those on Earth--that is, predict the number of stars besides our sun which might support planets suitable for human life.

References

It is not possible in a volume of this size to present a complete and comprehensive bibliography in all areas of science and mathematics. The list which follows is intended only to suggest some representative resources which contain material useful for developing activities similar to the ones described in this publication. In addition, basic high school and introductory college level texts provide background information on concepts and principles involved as well as data needed for some investigations. Supplementary publications and journal articles on selected topics are also extremely valuable references. Films and other audio-visual materials should not be overlooked as excellent resources as well.

Physical Science:

Abraham, N., et al. Interaction of Matter and Energy. Chicago: Rand McNally, 1968.

Engineering Concepts Curriculum Project. The Man-Made World. New York: McGraw Hill, 1971.

Haber-Schaim, Uri, et al. Introductory Physical Science Third edition. Englewood Cliffs: Prentice Hall,1977.

Intermediate Science Curriculum Study. The Natural World. Two volumes. Morristown, NJ: Silver Burdette, 1975.

Marean, John H., and Elaine W. Ledbetter. Physical Science: A Lab Approach. Menlo Park: Addison Wesley, 1968.

Nastasi, Richard, and Ernest Bacon. Physical Science--The Idea Science Program. Austin, TX: Steck-Vaughn, 1976.

Physical Science Group. Physical Science II. Englewood Cliffs: Prentice Hall, 1972.

PSNS Project Staff. An Approach to Physical Science. New York: Wiley, 1969.

Ramsey, et al. Holt Physical Science. New York: Holt, Rinehart and Winston, 1978.

Townsend, Ronald. Energy, Matter and Change. Glenview, IL: Scott, Foresman, 1973.

Physics:

Hewitt, Paul G. Conceptual Physics--A New Introduction to Your Environment. Third edition. Boston: Little, Brown, 1977.

Physical Science Study Committee. Physics. Boston: Heath, 1965.

Rutherford, F. James; Gerald Holton; and Fletcher Watson. The Project Physics Course. New York: Holt, Rinehart and Winston, 1970.

Williams, John E.; F. E. Trinklein; and H. Clark Metcalfe. Modern Physics. New York: 1976.

Chemistry:

Chemical Education Material Study. Chemistry, An Experimental Science. San Francisco: Freeman, 1963.

Cotton, F. A.; C. L. Darlington; and L. D. Lynch. Chemistry--An Investigative Approach. Boston: Houghton-Mifflin, 1976.

Darlington, C. L., and N. D. Eigenfeld. The Chemical World. Boston: Houghton-Mifflin, 1977.

Messer, M. B., et al. Introductory Experimental Chemistry. Englewood Cliffs: Prentice Hall, 1977.

O'Connor, et al. Chemistry: Experiments and Principles. Boston: Heath, 1968.

Weaver, Elbert. Scientific Experiments in Chemistry. New York: Holt, Rinehart and Winston,

Life Science:

Anderson, Norman D., and Walter R. Brown. Life Science--A Search for Understanding. Philadelphia: Lippincott, 1971.

Biological Sciences Curriculum Study. Biological Science: An Ecological Approach. Chicago: Rand McNally, 1973.

Biological Sciences Curriculum Study. Biological Science: An Inquiry into Life. New York: Harcourt, Brace, Jovanovich, 1973.

Biological Sciences Curriculum Study. Biological Science Interaction of Experiments and Ideas. Englewood Cliffs: Prentice Hall, 1965.

Biological Sciences Curriculum Study. Biological Science: Molecules to Man. Boston: Houghton-Mifflin, 1973.

Rasmussen, Frederick A.; Paul, Holabinko; and Victor M. Showalter. Man and the Environment. Boston: Houghton-Mifflin, 1971.

Earth Science:

Bernstein, L., and H. K. Wong. *Ideas and Investigations in Science--Earth Science*. Englewood Cliffs: Prentice Hall, 1977.

Earth Science Curriculum Project. *Investigating the Earth*. Boston: Houghton-Mifflin, 1967.

Ramsey, et al. *Holt Earth Science*. New York: Holt, Rinehart and Winston, 1978.

Wolfe, C. Wroe, et al. *Earth and Space Science*. Boston: Heath, 1966.

Mathematics:

Bakst, Aaron. *Mathematics, Its Magic and Mastery*. Princeton: Van Nostrand, 1952.

Bergamini, David. *Mathematics*. New York: Time-Life, 1972.

Fremont, Herbert. *Teaching Secondary Mathematics Through Applications*. Second edition. Boston: Prindle, Weber, and Schmidt, 1979.

Gardner, Martin. *Sixth Book of Mathematical Games*. New York: Scribner, 1971.

Ghyka, Matila. *The Geometry of Art and Life*. New York: Dover, 1977.

Gillespie, N. J. *MIRA Activities for Junior High School Geometry*. Palo Alto: Creative Publications, 1973.

Johnson, Donovan A. *Excursions in Outdoor Measurement*. Portland, ME: J. Weston Walch, 1974.

Menninger, Karl W. *Mathematics in Your World*. New York: Viking, 1962.

MIRA Math for Elementary School. Palo Alto: Creative Publications, 1973.

Mosteller, Frederick, et al. (eds.). *Statistics by Example*. 4 volumes. Menlo Park: Addison-Wesley, 1973.

National Aeronautics and Space Administration. *Space Mathematics, A Resource for Teachers*. Washington, DC: NASA, 1972.

Schaaf, William L. *Mathematics and Science: An Adventure in Postage Stamps*. Reston, VA: National Council of Teachers of Mathematics, 1978.

School Mathematics Study Group. *Mathematics Through Science*. Three volumes. Stanford University, 1963-64.

Schuh, Fred. *The Master Book of Mathematical Recreations*. New York: Dover, 1968.

Scientific American. *Mathematics in the Modern World*. San Francisco: Freeman, 1968.

Stover, Donald W. *Stereograms*. Boston: Houghton-Mifflin, 1966.

Tanur, Judith M., et al. (eds.). *Statistics: A Guide to the Unknown*. San Francisco: Holden-Day, 1972.

Usiskin, Zalman. *Algebra Through Applications*. Two volumes. Reston, VA: National Council of Teachers of Mathematics, 1979.

Vergara, William C. *Mathematics in Everyday Things.* New York: New American Library, 1962.